Navier-Stokes Equations in Spherical Coordinates

In spherical coordinates, (r, θ, ϕ), the Navier-Stokes equations of motion for an incompressible fluid with uniform viscosity are:

$$\rho\left[\frac{Du_r}{Dt}-\frac{u_\theta^2+u_\phi^2}{r}\right]=-\frac{\partial p}{\partial r}+f_r+\mu\left[\nabla^2 u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right] \tag{Bhh1}$$

$$\rho\left[\frac{Du_\theta}{Dt}+\frac{u_\theta u_r}{r}-\frac{u_\phi^2\cot\theta}{r}\right]=-\frac{1}{r}\frac{\partial p}{\partial \theta}+f_\theta+\mu\left[\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2\sin\theta\sin\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right] \tag{Bhh2}$$

$$\rho\left[\frac{Du_\phi}{Dt}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi\cot\theta}{r}\right]=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+f_\phi+\mu\left[\nabla^2 u_\phi+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r^2\sin\theta\sin\theta}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right] \tag{Bhh3}$$

where u_r, u_θ, u_ϕ are the velocities in the r, θ, ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{Bhh4}$$

and the Laplacian operator is

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin\theta\sin\theta}\frac{\partial^2}{\partial\phi^2} \tag{Bhh5}$$

Moreover, for an incompressible fluid the equation of continuity in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}=0 \tag{Bhh6}$$

and for an incompressible, Newtonian fluid

$$\sigma_{rr}=-p+2\mu\frac{\partial u_r}{\partial r}\quad;\quad\sigma_{\theta\theta}=-p+2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right) \tag{Bhh7}$$

$$\sigma_{\phi\phi}=-p+2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}\right) \tag{Bhh8}$$

$$\sigma_{r\theta}=\mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\quad;\quad\sigma_{r\phi}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right) \tag{Bhh9}$$

$$\sigma_{\theta\phi}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right)\right) \tag{Bhh10}$$