Transport Theorem

Here we derive a fundamental relation known as the Transport Theorem that relates the Lagrangian rate of change of the total amount of some transportable fluid property in a Lagrangian volume to the rate of change of that same quantity in an Eulerian volume that coincides with that Lagrangian volume at the time under consideration. Consider the Eulerian volume, V, of fluid shown by the red dashed line in figure 1. Next we consider the Lagrangian fluid volume that coincides with V at the initial time $t = 0$ under consideration. Though these volumes coincide at $t = 0$, the difference is that, being an Eulerian volume, V remains in the same location for all time, whereas, because of the fluid flow, the Lagrangian volume moves on with the flow and occupies a different location when $t \neq 0$. We denote the position of the Lagrangian volume at some small time later, $t = \delta t$, by V^* as shown by the blue dashed line in figure 1. For convenience we also denote the surface of V by S.

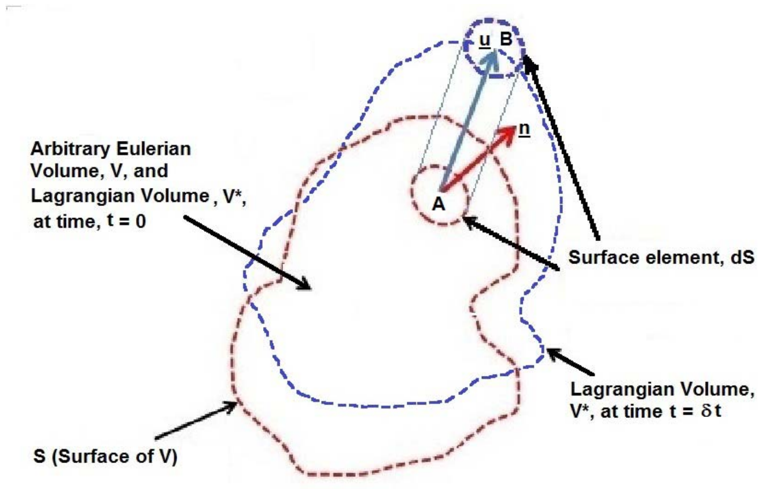

Figure 1: Arbitrary Eulerian Volume, V, and coincident Lagrangian Volume, V^*.

It follows that a point A on the surface of V where the velocity at $t = 0$ is denoted by the vector $\underline{u}$ will be displaced to the point B at $t = \delta t$ where the vector AB is equal to $\underline{u}\delta t$. Therefore, if we define a small area of the surface of V around the point A by dS, the volume of the parallelopiped swept out by dS between the times $t = 0$ and $t = \delta t$ will be

$$(\underline{u}\delta t dS)\ .\underline{n} = (\underline{u}.\underline{n})\delta t dS \tag{Bae1}$$

where $\underline{n}$ is the outward unit normal to the surface S at A.

Now consider the Lagrangian and Eulerian time derivatives of some general transportable property per unit volume in the fluid motion that we will denote by Q. The total amount of Q in the volume V is then given by the integral

$$\int_V Q dV \tag{Bae2}$$

The Lagrangian rate of change of time of the total amount of Q in the Lagrangian volume must therefore be given by

$$\frac{D}{Dt}\left\{\int_V Q dV\right\} = \left[\frac{\left\{\int_{V^*}\{Q\}_{t=\delta t} dV^* - \int_V \{Q\}_{t=0} dV\right\}}{\delta t}\right]_{\delta t \to 0} \tag{Bae3}$$

Notice in the numerator on the right hand side that the first and second terms have different integrands *and* different limits of integration. To progress we divide the first term into an integral over the volume V plus an integral over the small volume in between the volumes V and V^*:

$$\frac{D}{Dt}\left\{\int_V QdV\right\} = \left[\frac{\left\{\int_V \{Q\}_{t=\delta t}dV + \int_{V^*-V}\{Q\}_{t=\delta t}d(V^*-V) - \int_V \{Q\}_{t=0}dV\right\}}{\delta t}\right]_{\delta t\to 0} \tag{Bae4}$$

$$\frac{D}{Dt}\left\{\int_V QdV\right\} = \left[\frac{\left\{\int_V \left(\{Q\}_{t=\delta t} - \{Q\}_{t=0}\right)dV\right\}}{\delta t} + \frac{\left\{\int_{V^*-V}\{Q\}_{t=\delta t}d(V^*-V)\right\}}{\delta t}\right]_{\delta t\to 0} \tag{Bae5}$$

Examine the first term in the large square brackets. Since the Eulerian volume V does not change with time the δt in the denominator can be taken inside the integral. Turning to the second term the integral over the slender volume between V^* and V can be written as the integral over the surface area S of the incremental volume $(\underline{u}.\underline{n})\delta t dS$ times the value of Q at that location (whether we use $\{Q\}_{t=\delta t}$ or $\{Q\}_{t=0}$ does not matter because the difference disappears as $\delta t \to 0$). Thus the above expression becomes

$$\frac{D}{Dt}\left\{\int_V QdV\right\} = \left[\int_V \frac{(\{Q\}_{t=\delta t} - \{Q\}_{t=0})}{\delta t}dV + \int_S \frac{\{\{Q\}_{t=0}(\underline{u}\ .\underline{n})\delta t dS\}}{\delta t}\right]_{\delta t\to 0} = \int_V \frac{\partial Q}{\partial t}\ dV + \int_S Q(\underline{u}.\underline{n})\ dS \tag{Bae6}$$

Finally, using Gauss' theorem which states that for any vector field $\underline{q}$:

$$\int_S (\underline{q}.\underline{n})\ dS = \int_V \nabla.\underline{q}\ dV \tag{Bae7}$$

it follows that

$$\frac{D}{Dt}\left\{\int_V QdV\right\} == \int_V \frac{\partial Q}{\partial t}\ dV + \int_V \nabla.(Q\underline{u})\ dV = \int_V \frac{\partial Q}{\partial t} + \nabla.(Q\underline{u})\ dV \tag{Bae8}$$

This is the transport theorem. It is most valuable in expressing the Lagrangian rate of change of many different integral, transportable properties in terms of Eulerian quantities.