# Some Simple Transfer Matrices

The flow of an incompressible fluid in a straight, rigid pipe will be governed by the following versions of equations (Bnfb1) and (Bnfb2):

$$\frac{\partial u}{\partial s} = 0 \tag{Bngg1}$$

$$\frac{\partial p^T}{\partial s} = -\frac{\rho f u|u|}{4a} - \rho\frac{\partial u}{\partial t} \tag{Bngg2}$$

If the velocity fluctuations are small compared with the mean velocity denoted by $U$ (positive in direction from inlet to discharge), and the term $u|u|$ is linearized, then the above equations lead to the transfer function

$$[T] = \begin{bmatrix} 1 & -(R + j\omega L) \\ 0 & 1 \end{bmatrix} \tag{Bngg3}$$

where $(R + j\omega L)$ is the "impedance" made up of a "resistance", $R$, and an "inertance", $L$, given by

$$R = \frac{fU\ell}{2aA} \quad , \quad L = \frac{\ell}{A} \tag{Bngg4}$$

where $A$, $a$, and $\ell$ are the cross-sectional area, radius, and length of the pipe. A number of different pipes in series would then have

$$R = Q\sum_i \frac{f_i\ell_i}{2a_iA_i^2} \quad ; \quad L = \sum_i \frac{\ell_i}{A_i} \tag{Bngg5}$$

where $Q$ is the mean flow rate. For a duct of non-uniform area

$$R = Q\int_0^\ell \frac{f(s)ds}{2a(s)(A(s))^2} \quad ; \quad L = \int_0^\ell \frac{ds}{A(s)} \tag{Bngg6}$$

Note that all such ducts represent *reciprocal* and *symmetric* components.

A second, common hydraulic element is a simple "compliance", exemplified by an accumulator or a surge tank. It consists of a device installed in a pipeline and storing a volume of fluid, $V_L$, which varies with the local pressure, $p$, in the pipe. The compliance, $C$, is defined by

$$C = \rho\frac{dV_L}{dp} \tag{Bngg7}$$

In the case of a gas accumulator with a mean volume of gas, $\bar{V}_G$, which behaves according to the polytropic index, $k$, it follows that

$$C = \rho\bar{V}_G/k\bar{p} \tag{Bngg8}$$

where $\bar{p}$ is the mean pressure level. In the case of a surge tank in which the free surface area is $A_S$, it follows that

$$C = A_S/g \tag{Bngg9}$$

The relations across such compliances are

$$\tilde{m}_2 = \tilde{m}_1 - j\omega C\tilde{p}^T \quad ; \quad \tilde{p}_1^T = \tilde{p}_2^T = \tilde{p}^T \tag{Bngg10}$$

Therefore, using the definition (Bngc6), the transfer function $[T]$ becomes

$$[T] = \begin{bmatrix} 1 & 0 \\ -j\omega C & 1 \end{bmatrix} \tag{Bngg11}$$

Again, this component is *reciprocal* and *symmetric*, and is equivalent to a capacitor to ground in an electrical circuit.

Systems made up of lumped resistances, $R$, inertances, $L$, and compliances, $C$, will be termed LRC systems. Individually, all three of these components are both reciprocal and symmetric. Models of such systems are termed "Lumped Parameter Models" and these are frequently employed to analyze unsteady flows in internal flow systems. We note that any system comprised of these components will also be reciprocal (see the section on "Properties of Transfer Matrices"); hence all LRC systems are reciprocal. Note also that, even though individual components are symmetric, LRC systems are not symmetric since series combinations are not, in general, symmetric (see the section on "Properties of Transfer Matrices").

An even more restricted class of systems are those consisting only of inertances, $L$, and compliances, $C$. These systems are termed "dissipationless" and have some special properties (see, for example, Pipes 1963) though these are rarely applicable in hydraulic systems.

As a more complicated example, consider the frictionless ($f = 0$) compressible flow in a straight uniform pipe of mean cross-sectional area, $A_0$. This can readily be shown to have the transfer function

$$\begin{aligned} T^*_{11} &= (\cos\theta + jM\sin\theta)\, e^{j\theta M} \\ T^*_{12} &= -j\bar{U}\sin\theta e^{j\theta M}/A_0 M \\ T^*_{21} &= -jA_0 M(1-M^2)\sin\theta e^{j\theta M}/\bar{U} \\ T^*_{22} &= (\cos\theta - jM\sin\theta)\, e^{j\theta M} \end{aligned} \tag{Bngg12}$$

where $\bar{U}$ is the mean fluid velocity, $M = \bar{U}/c$ is the Mach number, and $\theta$ is a reduced frequency given by

$$\theta = \omega\ell/c(1-M^2) \tag{Bngg13}$$

Note that all the usual acoustic responses can be derived quite simply from this transfer function. For example, if the pipe opens into reservoirs at both ends, so that appropriate inlet and discharge conditions are $\tilde{p}_1 = \tilde{p}_2 = 0$, then the transfer function, equation (Bngc6), can only be satisfied with $\tilde{m}_1 \neq 0$ if $T^*_{12} = 0$. According to equations (Bngg12), this can only occur if $\sin\theta = 0$, $\theta = n\pi$ or

$$\omega = n\pi c(1-M^2)/\ell \tag{Bngg14}$$

which are the natural organ-pipe modes for such a pipe. Note also that the determinant of the transfer matrix is

$$D_T = D_{T^*} = e^{2j\theta M} \tag{Bngg15}$$

Since no damping has been included, this component is an undamped distributed system, and is therefore quasi-reciprocal. At low frequencies and Mach numbers, the transfer function (Bngg12) reduces to

$$\begin{aligned} &T^*_{11} \to 1 \quad ; \quad T^*_{12} \to -\frac{j\omega\ell}{A_0} \\ &T^*_{21} \to -j\left(\frac{A_0\ell}{c^2}\right)\omega \quad ; \quad T^*_{22} \to 1 \end{aligned} \tag{Bngg16}$$

and so consists of an inertance, $\ell/A_0$, and a compliance, $A_0\ell/c^2$.

When friction is included (as is necessary in most water-hammer analyses) the transfer function becomes

$$T_{11}^* = \left(k_1 e^{k_1} - k_2 e^{k_2}\right) / \left(k_1 - k_2\right)$$
$$T_{12}^* = -\bar{U}(j\theta + f^*)\left(e^{k_1} - e^{k_2}\right) / A_0 M \left(k_1 - k_2\right)$$
$$T_{21}^* = -j\theta A_0 M(1 - M^2)\left(e^{k_1} - e^{k_2}\right) / \bar{U}\left(k_1 - k_2\right)$$
$$T_{22}^* = \left(k_1 e^{k_2} - k_2 e^{k_1}\right) / \left(k_1 - k_2\right) \quad \text{(Bngg17)}$$

in which $f^* = f\ell M/2a(1 - M^2)$ and $k_1, k_2$ are the solutions of

$$k^2 - kM(2j\theta + f^*) - j\theta(1 - M^2)(j\theta + f^*) = 0 \quad \text{(Bngg18)}$$

The determinant of this transfer matrix $[T^*]$ is

$$D_{T^*} = e^{k_1 + k_2} \quad \text{(Bngg19)}$$

Note that this component is only quasi-reciprocal in the undamped limit, $f \to 0$.