Law of the Wall

Another valuable tool in understanding and analyzing turbulent flows is obtained by dimensional analysis of the flow near a solid boundary. It is known as the *Law of the Wall* and is derived by assuming that the turbulence near that boundary is a function only of the flow conditions pertaining at that wall and is independent of the flow conditions further away. For the non-dimensional analysis we can identify the following limited set of conditions and fluid properties (Figure 1):

- The distance, y, from the wall $[L]$.
- The mean velocity (or velocity profile), $\overline{u}(y)$ $[L/T]$.
- The shear stress, τ_W, at the wall $[M/LT^2]$.
- The fluid density, ρ $[M/L^3]$.
- The fluid kinematic viscosity, ν $[L^2/T]$.

where the dimensions (mass, $[M]$, length, $[L]$, and time, $[T]$) of each quantity are indicated in square brackets and the density and viscosity are assumed uniform and constant. It is convenient to introduce a quantity called the *friction velocity*, u_τ, defined by

$$u_\tau = \left(\frac{\tau_W}{\rho}\right)^{\frac{1}{2}} \quad \text{(Bkj1)}$$

and dimensional analysis then yields only two dimensionless quantities namely

$$\text{A dimensionless length,} \quad y^* = \frac{y}{\nu}\left(\frac{\tau_W}{\rho}\right)^{\frac{1}{2}} = \frac{u_\tau y}{\nu}$$

$$\text{and a dimensionless velocity,} \quad u^* = \overline{u}\left(\frac{\tau_W}{\rho}\right)^{-\frac{1}{2}} = \frac{\overline{u}}{u_\tau} \quad \text{(Bkj2)}$$

Absent any other conditions, properties or influences it must follow that u^* is a function only of y^*. In practice, of course, boundaries or conditions further away must begin to have an effect on u^* at large y^* but leave that for later consideration.

It therefore seems valuable to try to identify the function $u^*(y^*)$, the so-called *universal velocity profile* for turbulent flow near a wall. First and most obviously a laminar flow near the wall will necessarily be a simple shear flow in which

$$\tau_w = \rho\nu\frac{\partial u}{\partial y} \quad \text{(Bkj3)}$$

which with the present notation translates simple to

$$u^* = y^* \quad \text{(Bkj4)}$$

Moreover, in a turbulent flow since the turbulent fluctuations must go to zero at the wall it follows that there will always be a very small layer next to the wall in which the flow is essentially laminar and in which $u^* = y^*$. This is called the *laminar sub-layer* and is present whatever the flow outside it may be. The only question is how deeply do significant turbulent perturbations penetrate this laminar sublayer and thus determine its thickness.

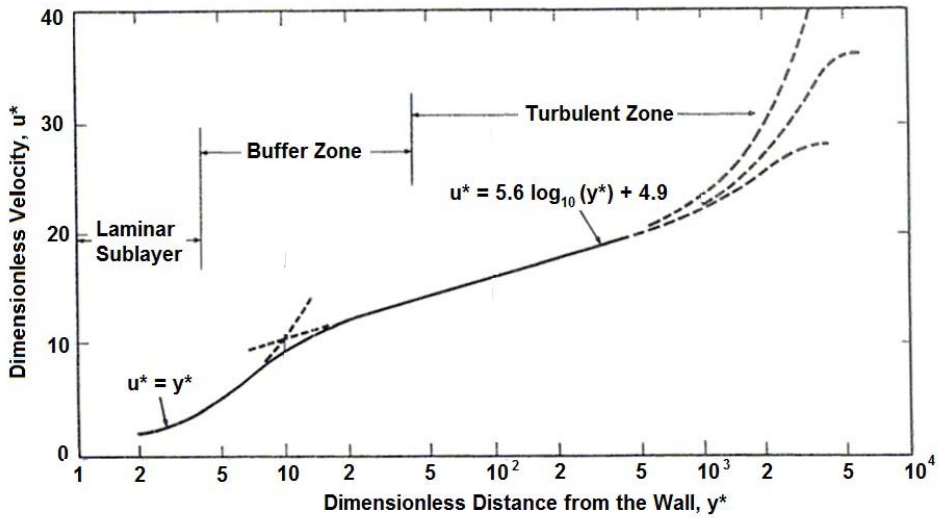

Figure 1: The universal velocity profile.

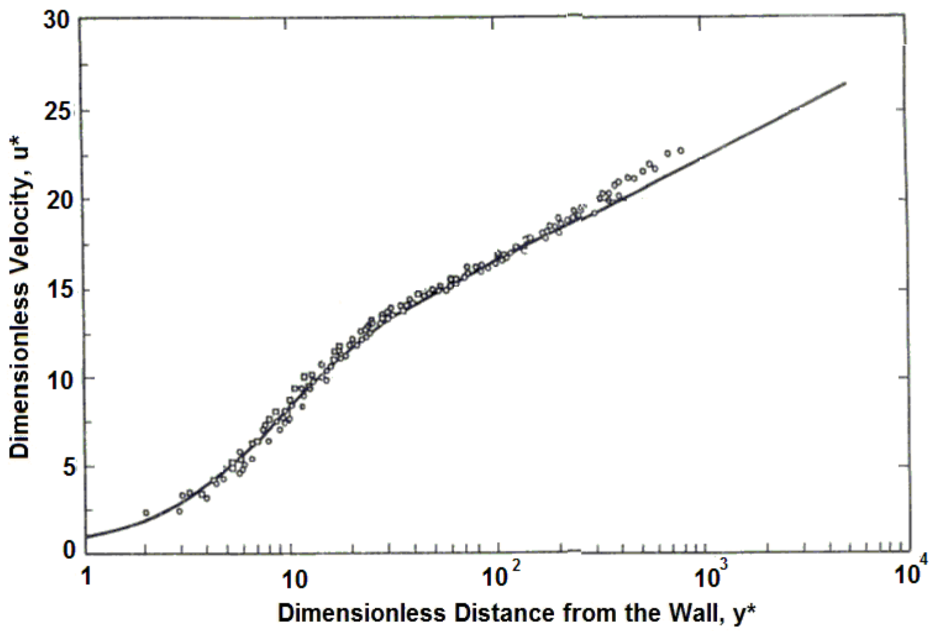

Figure 2: Comparison of experimental data with the universal velocity profile.

Further from the wall where the turbulent fluctuations dominate we require some model of the turbulence to construct the function $u^*(y^*)$. As an instructive example, we will use Prandtl's mixing length model to write:

$$\frac{\tau_W}{\rho} \;=\; -\overline{u'v'} \;\approx\; \kappa^2 y^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 \tag{Bkj5}$$

and, assuming some constant value for κ this integrates to

$$u^* \;=\; \frac{1}{\kappa}\ln y^* + C \;=\; \frac{2.303}{\kappa}\; log_{10}(y^*) + C \tag{Bkj6}$$

where C is an integration constant. It remains therefore to determine the integration constant, C, and to match the velocity profile in the laminar sublayer, $u^* = y^*$ to that in the turbulent region further away from the wall; this has been done by comparing with experimental data in order to generate a "universal" velocity profile of the turbulent flow near a solid boundary. The following are the typical results for such a "universal" velocity profile:

- The laminar sublayer in which $u^* = y^*$ extends out to about $y^* = 5$ whic in terms of the dimensional quantities means that the laminar sub-layer thickness, δ_{LSL}, is given approximately by

$$\delta_{LSL} = 5\nu/(\tau_W/\rho)^{\frac{1}{2}} \tag{Bkj7}$$

- There is a transitional "buffer zone" that extends from $y^* = 5$ to about $y^* = 30$.
- The fully turbulent flow begins about $y^* = 30$ and exhibits a velocity profile given by

$$u^* = C_1 \; log_{10}(y^*) + C_2 \tag{Bkj8}$$

  where the constant C_1 lies between about 5.6 and 5.75 and the constant C_2 lies between about 4.9 and 5.5. It transpires that this turbulent profile extends out into the flow much further than one might expect; it other words the modifications caused by other boundaries or flow features are smaller than one would expect. For example, in the analysis of turbulent pipe flow (section (Bkl)) this profile is successfully used all the way to the center of the pipe.

This universal velocity profile is sketched in Figure 1 and compared with experimental data in Figure 2.

There are two important footnotes to this summary of the "universal" velocity profile. First note that the above applies to a *smooth* wall though it will remain to be seen what constitutes a smooth wall. That question and the universal velocity profile for a rough wall are both detailed in later sections. Second, although the profile detailed above might be an accurate fit to the experimental data, its functional form may not be the most convenient for use in some applications. Therefore the following simpler but cruder form of the velocity profile,

$$u^* = 8.7(y^*)^{\frac{1}{7}} \tag{Bkj9}$$

is also widely used. Known as the *one-seventh power law* profile it was suggested by Blasius.

The above version of the law of the wall (and its universal profile) applies when the solid surface is smooth and has no other dimensions or features that should be included in the dimensional analysis (we will see below how smooth the wall has to be for that to be the case). Now we consider the other alternative in which the wall is so rough that the flow has no laminar sublayer; instead the flow over the roughness produces a turbulent flow next to the surface. We denote the size of the roughnesses by ϵ; this can and is related to the size of the sandgrain roughnesses employed in experiments but the detail of that relation is not needed here. Because of the roughness dimension, ϵ, we must add this to the list of properties that we compiled at the beginning of this section and this clearly leads to the addition of another dimensionless quantity, namely y/ϵ. To pursue the dimensional analysis further we argue that since the laminar sublayer is no more, there is no zone in the flow in which viscosity is consequential and therefore we might drop the viscosity from the list of properties that should be included (we need to return at the end of these deliberations and validate that assumption). Without viscosity the dimensional analysis leads simply to the conclusion that the dimensionless velocity, $u^* = \overline{u}/u_\tau$, can only be a function of y/ϵ. As an approximate evaluation of the domain of validity of this flow regime we recall that the laminar sublayer thickness in the smooth regime, δ_{LSL}, is given by equation (Bkj7) and it seems reasonable to suggest that the fully rough regime with occur when

$$\epsilon \ll \delta_{LSL} \qquad \text{or} \qquad \frac{\epsilon(\tau_W/\rho)^{\frac{1}{2}}}{\nu} \ll 5 \tag{Bkj10}$$

Of course, there will be an intermediate or transitional regime between the smooth and fully-rough cases.

To obtain the form of the universal velocity profile for the fully-rough case, we follow a procedure parallel to that used earlier in the smooth case and, using the Prandtl mixing length model and integrating, it transpires that the universal velocity profile for the fully-rough regime is necessarily of the form

$$u^* \; = \; \frac{\overline{u}}{u_\tau} \; = \; \frac{1}{\kappa} \ln \left\{ \frac{y}{\epsilon} \right\} + C \tag{Bkj11}$$

where C is some integration constant. Comparison with experiment yields a universal velocity profile given by

$$u^* \; = \; \frac{\overline{u}}{u_\tau} \; = \; C_3 \; log_{10}(y/\epsilon) + C_4 \tag{Bkj12}$$

where estimated values of C_3 and C_4 are approximately 5.75 and 8.5 respectively.