Singularities in Stokes' Flow

To repeat, the basic equations for the flow of a viscous, incompressible flow when the inertia terms are neglected consist of a continuity condition on the fluid velocity $\underline{u}$,

$$\nabla.\underline{u} = 0 \tag{Blc1}$$

and since there are no inertial forces, a condition of force equilibrium

$$\nabla p = \mu\nabla^2\underline{u} \tag{Blc2}$$

where the dynamic viscosity is denoted by μ and p denotes the fluid pressure. From this it follows that p is a harmonic function, and therefore the velocity is a bi-harmonic function. Most importantly these equations are linear and therefore simple solutions can be superimposed to construct almost any three-dimensional flow at zero Reynolds number. The most basic or fundamental of these solutions, the most basic building block, is that generated by a single point force, $\underline{F}$, in an unbounded (and inertialess) fluid. It was first obtained by Oseen (1927), developed further by Burgers (1938), and named a *stokeslet* by Hancock (1953). If one represents the strength and direction of the singular force at the origin of a coordinate system $\underline{x}$ by $8\pi\mu\underline{\alpha}$, where $\underline{\alpha}$ denotes the stokeslet strength and direction, the resulting fluid velocity and pressure are, respectively, (see, for example, Chwang & Wu 1975)

$$\underline{u} = \underline{\alpha}/r + (\underline{\alpha}.\underline{x})\underline{x}/r^3 \qquad \text{and} \qquad p = 2\mu(\underline{\alpha}.\underline{x})/r^3 \tag{Blc3}$$

where $r = |\underline{x}|$. Due to the linearity it also follows that a derivative of any order of this solution is also a solution to the basic equations. Thus one can construct higher-order singularities such as a Stokes doublet, Stokes quadrupole, etc. Batchelor (1970b) indicated how a Stokes doublet could be decomposed into an antisymmetric component representing the flow field due to a singular moment of strength and orientation $8\pi\mu\underline{\gamma}$ and called a couplet [Chwang & Wu (1974) call this a rotlet] with velocity and pressure

$$\underline{u} = \underline{\gamma}\times\underline{x}/r^3 \qquad \text{and} \qquad p = 0 \tag{Blc4}$$

and a symmetric component representing a pure straining or extensional motion of the fluid and termed a stresslet. Furthermore, a Laplacian of the stokeslet solution leads to a potential doublet of strength and orientation $\underline{\delta}$ for which

$$\underline{u} = -\underline{\delta}/r^3 + 3(\underline{\delta}.\underline{x})\underline{x}/r^5 \qquad \text{and} \qquad p = 0 \tag{Blc5}$$

and which has zero vorticity. One sees that this has the same kinematic form as the conventional doublet in potential flow of an inviscid fluid but that its dynamic contribution to pressure is now zero because the inertia terms have been deleted.

Chwang & Wu (1974, 1975) and Chwang (1975) have shown how solutions to many complex flows may be constructed by superposition of these fundamental singularities; indeed, for mathematically simple bodies such as spheroids in mathematically simple flow fields (uniform flow, shear flow, quadratic flow, extensional flow, etc.) exact solutions are obtained. The simplest example is that of rectilinear translation (at velocity U) of a sphere of radius a, which requires at the center of the sphere only a stokeslet of strength $3aU/4$ in the forward direction and a potential doublet of strength $a^3U/4$ in the opposite direction in order to satisfy the no-slip boundary condition at the surface of the sphere. Indeed, one can visualize the stokeslet as simulating the drag on the body; this is the dominating effect in the far-field since a stokeslet, being the lowest-order singularity, decays least rapidly (like $1/r$). Furthermore, the potential doublet provides the finite geometry of the body in the near-field and its velocity contribution is necessary to satisfy the no-slip condition at the body surface (see Lighthill 1975, p. 48).

We reiterate Chwang & Wu's (1975) observation on the exact solution for the translation of a prolate spheroid of major axis, a, and minor axis, b. They observed that if the translational velocity is decomposed into components U_s and U_n, parallel and perpendicular to the major axis and if one examined the force on an element of this spheroid contained between two planes perpendicular to the major axis and length ds apart, then this was composed of two components F_s and F_n, in the same two directions where

$$F_s = -C_s U_s ds \qquad \text{and} \qquad F_n = -C_n U_n ds \tag{Blc6}$$

and C_s and C_n were simple constants dependent only on μ, a and b and independent of the position of the element or the velocities U_s and U_n. This is a remarkable example of a case in which the resistive-force theory that we examine later holds exactly, irrespective of the slenderness of the body. For a slender prolate spheroid such that $b/a = \epsilon \ll 1$, the resistive coefficients, C_s and C_n, become

$$C_s = \frac{2\pi\mu}{\ln\frac{2a}{b} - \frac{1}{2}}\left[1 + O(\epsilon^2)\right] \tag{Blc7}$$

$$C_n = \frac{4\pi\mu}{\ln\frac{2a}{b} + \frac{1}{2}}\left[1 + O(\epsilon^2)\right] \tag{Blc8}$$

It is wise to note at this point that the solutions above represent exact solutions only at zero-Reynolds number. Introduction of the small contribution of inertia at low but finite Reynolds numbers necessitates re-examination of the far-field where the magnitude of the inertia terms becomes comparable with the viscous terms and leads, for example, to the well-known Stokes paradox for translation of an infinitely long cylinder. However linearization of these far-field inertia effects by means of Oseens approximation permits the construction of flow fields by means of a modified set of fundamental solutions in which the oseenlet replaces the stokeslet. Developments of slender-body theory along these lines have been proposed by Chwang & Wu (1976). Finally it should also be noted that there exists the fundamental solution of the entire Navier-Stokes equations for a singular force known as the "round laminar jet" and due to Slezkin (1934), Landau (1944), and Squire (1951). The stokeslet is simply the limiting case of the round laminar jet for an inertialess fluid. However the nonlinearity of the Navier-Stokes equations does not permit superposition of these fundamental solutions.

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In the proximity of a boundary, whether it is a solid wall, a free surface, or a hypothetical boundary simulating a line of symmetry in the flow under consideration, it becomes advantageous to develop image systems for the fundamental singularities constructed so that the boundary conditions on that wall are automatically satisfied. In inviscid potential flow this is usually a simple matter since, for example, a solid plane boundary requires only the identical singularity at the image point in order to satisfy the condition of zero normal velocity. However, at low Reynolds numbers one must also satisfy the no-slip condition, and the types of singularity required at the image point in order to accomplish this are not immediately obvious. Blake (1971c) obtained the image system for a stokeslet (at various orientations) in a stationary plane boundary, and Blake & Chwang (1974) derived similar image systems for a couplet, a source, and a potential doublet. Some of these are indicated schematically in Figure 1.

One of the important effects of the presence of the wall (or equivalently the image system) is that the nature of the far-field is altered. A stokeslet oriented parallel to a wall leads to a far-field, which is a stokes doublet decaying like r^{-2} rather than the r^{-1} of a stokeslet in unbounded fluid. On the other hand the far-field of a stokeslet oriented perpendicular to a wall is even weaker and is like a stokes quadrupole or potential

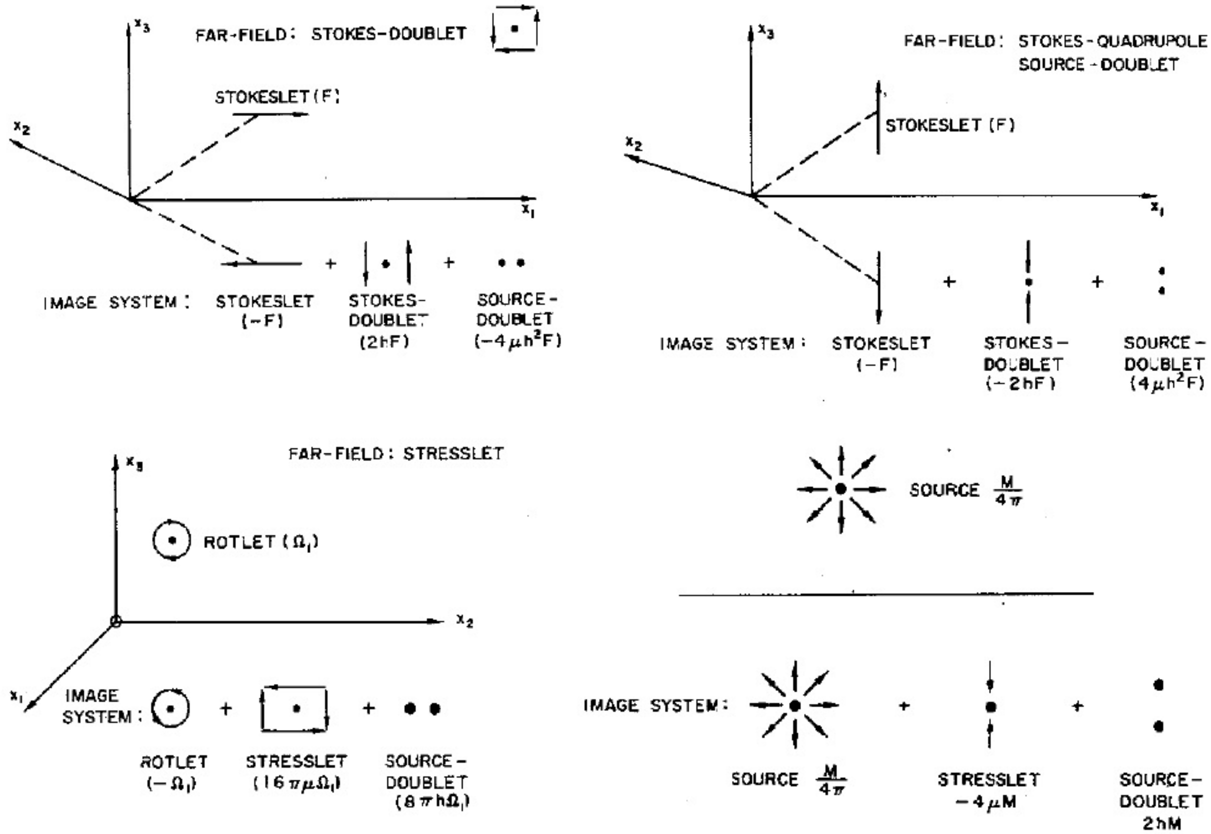

Figure 1: The image singularities for a no-slip boundary (here in the x_1x_2 plane) for a stokeslet tangential to the boundary (upper left), a stokeslet normal to the boundary (upper right), a rotlet whose axis is parallel to the boundary (lower left) and a source (lower right). From Brennen and Winet (1977).

doublet. Blake & Sleigh (1974) found that this has important consequences for the hydromechanics of cilia or for flagella near walls. The far-fields of the other singularities in the presence of a wall are similarly affected, the far-fields for both a rotlet and a source becoming stresslets (like r^{-2}); note that this differs from the far-field of a source near a wall in inviscid potential flow in the absence of a no-slip condition that is like a potential doublet (r^{-3}).