COUETTE AND PLANAR POISEUILLE FLOW

Couette and planar Poiseuille flow are both steady flows between two infinitely long, parallel plates a fixed distance, h, apart as sketched in Figures 1 and 2. The difference is that in Couette flow one of the plates

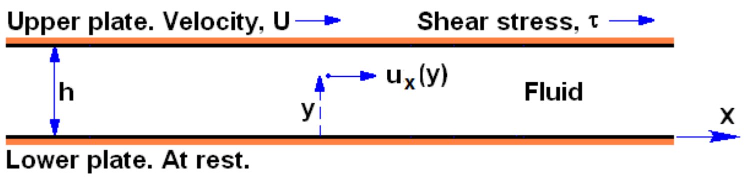

Figure 1: Couette flow.

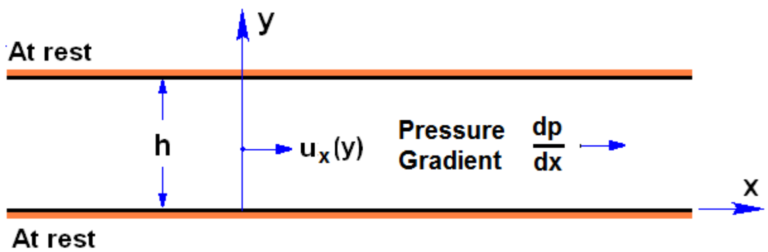

Figure 2: Planar Poiseuille flow.

has a velocity U in its own plane (the other plate is at rest) as a result of the application of a shear stress, τ, and there is no pressure gradient in the fluid. In contrast in planar Poiseuille flow both plates are at rest and the flow is caused by a pressure gradient, dp/dx, in the direction, x, parallel to the plates. It is however, convenient, to begin the analysis of these flows together. We will omit any conservative body forces like gravity since their effects are can be simply added to the final solutions. Then, assuming that the only non-zero component of the velocity is u_x and that the velocity and pressure are independent of time the resulting planar continuity equation for an incompressible fluid yields

$$\frac{\partial u_x}{\partial x} = 0 \qquad \text{(Bib1)}$$

so that $u_x(y)$ is a function only of y, the coordinate perpendicular to the plates. Using this the planar Navier-Stokes equations for an incompressible fluid of constant and uniform viscosity reduce to

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u_x}{\partial y^2} \qquad \text{(Bib2)}$$

$$\frac{\partial p}{\partial y} = 0 \qquad \text{(Bib3)}$$

The second of these shows that the pressure, $p(x)$, is a function only of x and hence the gradient, dp/dx, is well defined and a parameter of the problem. This allows the first of these equations (Bib2) to be integrated so that the velocity, u_x, can be written as

$$u_x = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{y^2}{2} + C_1 y + C_2 \qquad \text{(Bib4)}$$

where C_1 and C_2 are integration constants to be determined by the application of the boundary conditions at the two plates. Here the solutions for Couette flow and planar Poiseuille flow diverge.

Addressing first Couette flow for which $dp/dx = 0$ and applying the no-slip conditions at the upper and lower plates, namely

$$(u_x)_{y=h} \;=\; U \qquad \text{and} \qquad (u_x)_{y=0} \;=\; 0 \tag{Bib5}$$

yields

$$C_1 \;=\; \frac{U}{h} - \frac{C_2}{h} \qquad \text{and} \qquad C_2 \;=\; 0 \tag{Bib6}$$

and so the solution to Couette flow is

$$u_x \;=\; \frac{Uy}{h} \tag{Bib7}$$

and the shear stress at the walls is $\tau = \mu U/h$. Indeed a simple application of the momentum theorem to a rectangular control volume within the device will show that the shear stress, σ_{xy}, anywhere within the fluid is equal to $\mu U/h$.

Couette flow is frequently used to measure the viscosity of a fluid though, to avoid end effects, the flow is typically contained between two concentric cylinders as depicted in Figure 3. The radius of the cylinders

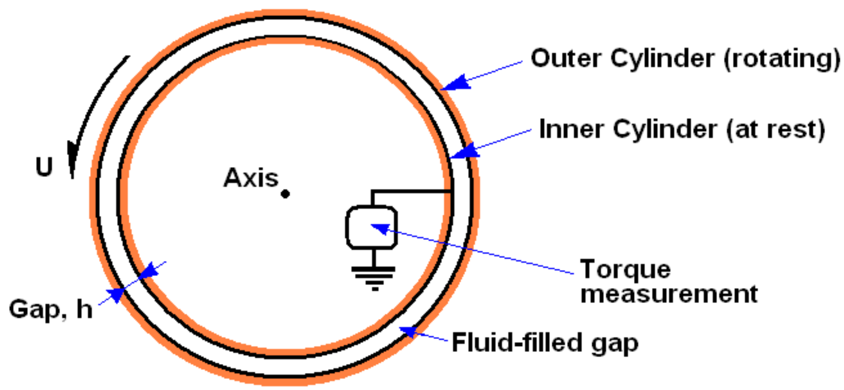

Figure 3: Couette viscometer.

must be large relative to the gap width, h, in order to avoid the effects of the curvature of the cylinders. By measuring the speed of rotation of the inner cylinder and the force required to hold the outer cylinder in place both U and τ can be evaluated and hence $\mu = \tau h/U$.

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The solution for planar Poiseuille flow proceeds along similar lines except, of course, that dp/dx is not zero. Applying the no-slip conditions at the lower and upper walls, namely

$$(u_x)_{y=0} \;=\; 0 \qquad \text{and} \qquad (u_x)_{y=h} \;=\; 0 \tag{Bib8}$$

yields

$$C_2 \;=\; 0 \qquad \text{and} \qquad C_1 \;=\; -\frac{h}{2}\left(\frac{dp}{dx}\right) \tag{Bib9}$$

and so the solution to planar Poiseuille flow is

$$u_x \;=\; \frac{1}{\mu}\left(-\frac{dp}{dx}\right)\frac{y}{2}(h-y) \tag{Bib10}$$

where the pressures, p_1 and p_2, could be measured at two different x locations a distance ℓ apart in order to determine $dp/dx = (p_1 - p_2)/\ell$. The velocity distribution in the fluid is parabolic with a maximum velocity on the centerline of

$$(u_x)_{y=h/2} = \frac{h^2}{8\mu}\left(-\frac{dp}{dx}\right) \tag{Bib11}$$

and the volume flow rate, $\dot{Q}$, per unit depth normal to the plane of the flow is

$$\dot{Q} = \int_0^h u_x \, dy = \frac{h^3}{12\mu}\left(-\frac{dp}{dx}\right) \tag{Bib12}$$

so that the average velocity of the flow, $\overline{u}$, is

$$\overline{u} = \frac{h^2}{12\mu}\left(-\frac{dp}{dx}\right) \tag{Bib13}$$

so the average is 2/3 of the maximum. The shear stresses, τ, at the walls are

$$\tau_{y=0} = \frac{h}{2}\left(-\frac{dp}{dx}\right) \quad \text{and} \quad \tau_{y=h} = -\frac{h}{2}\left(-\frac{dp}{dx}\right) \tag{Bib14}$$

and the shear stress within the fluid varies linearly between the plates according to

$$\sigma_{xy} = \left(-\frac{dp}{dx}\right)\left(\frac{h}{2} - y\right) \tag{Bib15}$$

We note, parenthetically, that these expressions (Bib14) and (Bib15) for the shear stresses can be derived using the momentum theorem applied to a simple rectangular control volume within the fluid. The results are independent of the viscosity and, in fact, independent of the constitutive properties of the fluid (or solid) contained between the two plates. Finally we should note that the above results for planar Poiseuille flow only have practical application up to Reynolds numbers, $\rho\overline{u}h/\mu$, of about 2000 for above that value the flow will transition from laminar to turbulent.