Planar Supersonic Flows for Small Deflections

In this section we analyze supersonic flows that involve small deflections in the direction of the flow, flows with small angles of turn. The context is displayed in Figure 1 where a flow with $M > 1$ is initially parallel with a flat wall but is forced to turn through a small angle, $d\theta$, because the wall makes that turn. The small angle, $d\theta$, can be positive or negative but will be defined as positive when the wall turns into the flow. The flow upstream of the turn is described by the velocity components, u and $v = 0$, parallel and normal

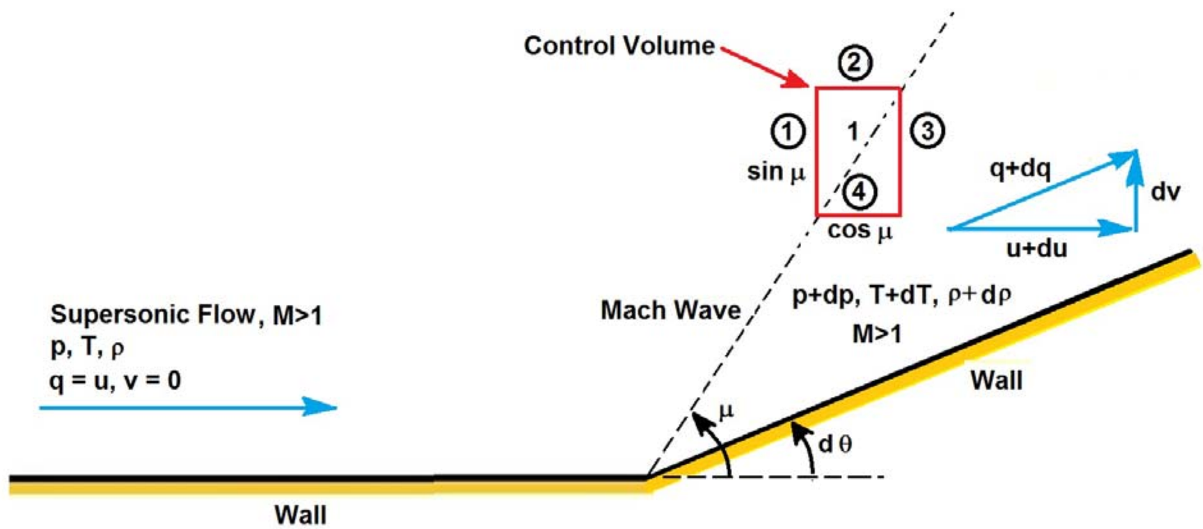

Figure 1: Notation for a supersonic flow performing a small angle of turn, $d\theta$.

to the upstream wall, and by the Mach number, M, pressure, p, density, ρ, and temperature, T. A Mach wave is generated where the wall turns and is inclined at the angle, μ, to the upstream wall. The angle of turn, $d\theta$, is small and so the flow downstream of the Mach wave is still supersonic with Mach number, $M + dM$, and is therefore parallel with the downstream wall and has a pressure, $p + dp$, density, $\rho + d\rho$, and temperature, $T + dT$. The velocity components of this downstream flow are denoted by $u + du$ and dv and the magnitude of this velocity is denoted by $q + dq$ where $q \approx u$. In order to apply the basic conservation equations we choose to utilize the control volume shown by the red box in Figure 1 whose diagonal has unit length and sides measuring $\sin\mu$ and $\cos\mu$ as shown in the Figure. For convenient reference the four sides of this control volume are labeled 1, 2, 3 and 4 as indicated in the figure; it also has unit dimension normal to the sketch.

The conservation equations applied to this control volume yield the following:

- *Continuity:*

$$\rho u \sin\mu \;=\; \left[(u + du)\sin\mu - dv\cos\mu\right](\rho + d\rho) \qquad \text{(Bok1)}$$

  and neglecting all terms quadratic in the small quantities this yields

$$u\sin\mu \; d\rho + \rho\sin\mu \; du - \rho\cos\mu \; dv \;=\; 0 \qquad \text{(Bok2)}$$

- *Momentum parallel to the upstream wall:*

$$-\sin\mu \; dp \;=\; (\rho + d\rho)\left[(u + du)^2\sin\mu - (u + du)\cos\mu \; dv\right] - \rho u^2\sin\mu \qquad \text{(Bok3)}$$

and neglecting all terms quadratic in the small quantities this yields

$$-\sin\mu\ dp \ = \ (u^2 d\rho + 2\rho u du)\sin\mu - \rho u\cos\mu\ dv \tag{Bok4}$$

- *Momentum normal to the upstream wall:*

$$\cos\mu\ dp \ = \ (\rho + d\rho)\left[(u+du)\sin\mu\ dv - (\rho+d\rho)\ dv\ dv\right] - \rho u^2 \sin\mu \tag{Bok5}$$

and neglecting all terms quadratic in the small quantities this yields

$$\cos\mu\ dp \ = \ \rho u \sin\mu\ dv \tag{Bok6}$$

Eliminating dp and $d\rho$ from equations (Bok2), (Bok4) and (Bok6), using the trigonometric relation $dv = u\ d\theta$ (neglecting quadratic terms) and the expression for $\sin\mu$:

$$\frac{du}{u} \ = \ \frac{dq}{q} \ = \ \tan\mu\ d\theta \ = \ \frac{d\theta}{(M^2-1)^{\frac{1}{2}}} \tag{Bok7}$$

Notice that when the wall turns into the flow ($d\theta > 0$), the velocity decreases and the pressure increases; this is known as a *compression turn*. On the other hand when the wall turns away from the flow ($d\theta < 0$), the velocity increases and the pressure decreases; this is known as an *expansion turn*.

In addition the above basic equations with the perfect gas law and the definition of the Mach number yield:

$$\frac{dp}{p} \ = \ \frac{\gamma M^2\ d\theta}{(M^2-1)^{\frac{1}{2}}} \quad ; \quad \frac{dT}{T} \ = \ \frac{(\gamma-1)M^2\ d\theta}{(M^2-1)^{\frac{1}{2}}} \quad ; \quad \frac{d\rho}{\rho} \ = \ \frac{M^2\ d\theta}{(M^2-1)^{\frac{1}{2}}} \tag{Bok8}$$

$$\frac{dM}{M} \ = \ -\frac{[1+(\gamma-1)M^2/2]}{(M^2-1)^{\frac{1}{2}}}\ d\theta \tag{Bok9}$$

We note that it has not been assumed that the change is isentropic; in fact, since $dp/p = \gamma d\rho/\rho$ it is, in fact, isentropic because the changes are very small.

A number of useful examples of solutions to supersonic flows involving small flow deflections will be developed.

Supersonic Flow past a flat plate at a small angle of attack:

A convenient and useful example of such a flow is an infinitely thin flat plate set at a small angle of attack (α) to an oncoming supersonic stream of Mach number, M, as sketched in Figure 2. The angle of turn from the upstream flow to the flow on the upper side of the plate (Region 2) is then $d\theta = -\alpha$ and therefore, according to the relation (Bok8), the pressure difference $p_2 - p_1$ is given by

$$p_2 - p_1 \ = \ -\frac{\gamma M_1^2 \alpha p_1}{(M_1^2-1)^{\frac{1}{2}}} \ = \ -\frac{\rho_1 q_1^2 \alpha}{(M_1^2-1)^{\frac{1}{2}}} \tag{Bok10}$$

On the other hand the angle of turn from the upstream flow to the flow on the lower side of the plate (Region 3) is $d\theta = \alpha$ and therefore, according to the relation (Bok8), the pressure difference $p_2 - p_1$ is given by

$$p_3 - p_1 \ = \ \frac{\gamma M_1^2 \alpha p_1}{(M_1^2-1)^{\frac{1}{2}}} \ = \ \frac{\rho_1 q_1^2 \alpha}{(M_1^2-1)^{\frac{1}{2}}} \tag{Bok11}$$

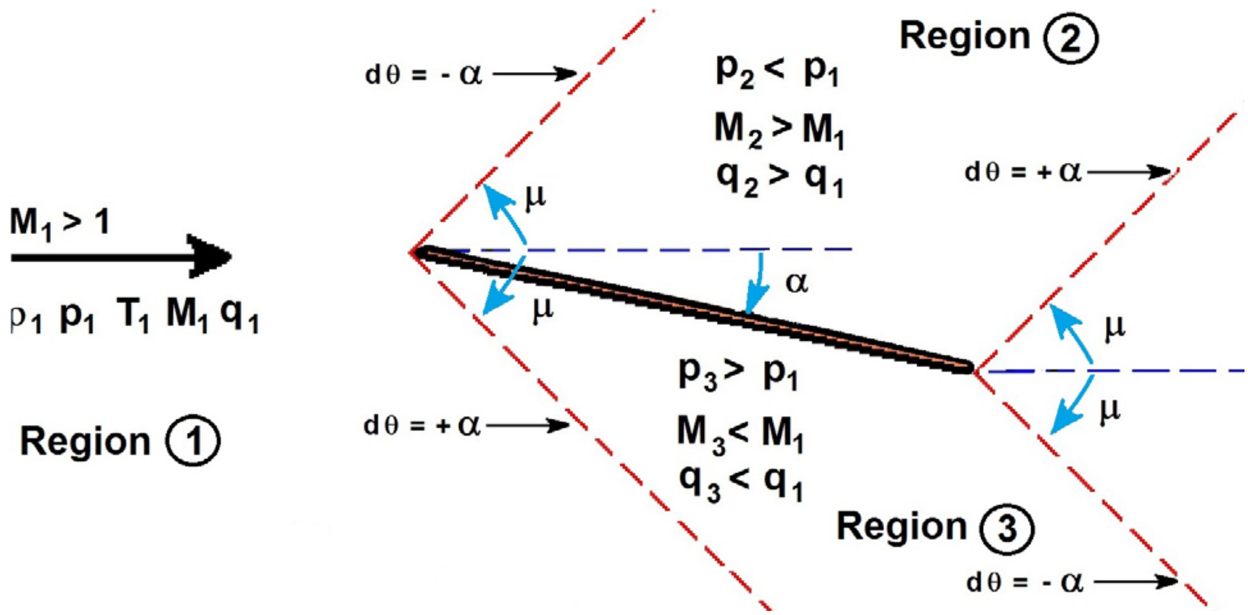

Figure 2: Supersonic flow past a flat plate at a very small angle of attack, α.

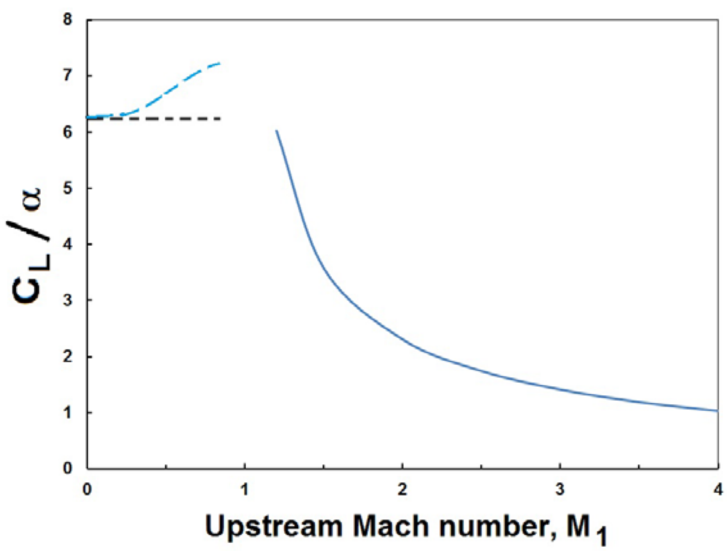

Figure 3: Flat plate lift/slope, C_L/α, plotted against the upstream Mach number, M_1.

Hence the pressure difference across the plate is

$$p_3 - p_2 \;=\; \frac{2\rho_1 q_1^2 \alpha}{(M_1^2-1)^{\frac{1}{2}}} \tag{Bok12}$$

and, to leading order in α, the resulting lift and drag forces acting on the plate, L and D, are

$$L \;=\; \frac{2\rho_1 q_1^2 \alpha A}{(M_1^2-1)^{\frac{1}{2}}} \qquad \text{and} \qquad D \;=\; \frac{2\rho_1 q_1^2 \alpha^2 A}{(M_1^2-1)^{\frac{1}{2}}} \tag{Bok13}$$

where A is the planform area of the plate. Using the normal definitions, the lift and drag coefficients, C_L and C_D, are

$$C_L \;=\; \frac{2L}{\rho_1 q_1^2 A} \;=\; \frac{4\alpha}{(M_1^2-1)^{\frac{1}{2}}} \qquad \text{and} \qquad C_D \;=\; \frac{2D}{\rho_1 q_1^2 A} \;=\; \frac{4\alpha^2}{(M_1^2-1)^{\frac{1}{2}}} \tag{Bok14}$$

Figure 3 is a plot of the lift/slope, C_L/α, for the flat plate plotted against the upstream Mach number, M_1; it includes the zero Mach number value of 2π.

Supersonic flow past a triangular airfoil at a small angle of attack:

The next example involves a minor modification to the first example, designed to demonstrate the effect of finite thickness on the performance of an airfoil in supersonic flow. For simplicity we will solve the flow sketched in Figure 4, an airfoil with a triangular cross-section where the thickness is represented by the angle, β. Then, using equation (Bok8) the pressures in regions 2, 3 and 4 become

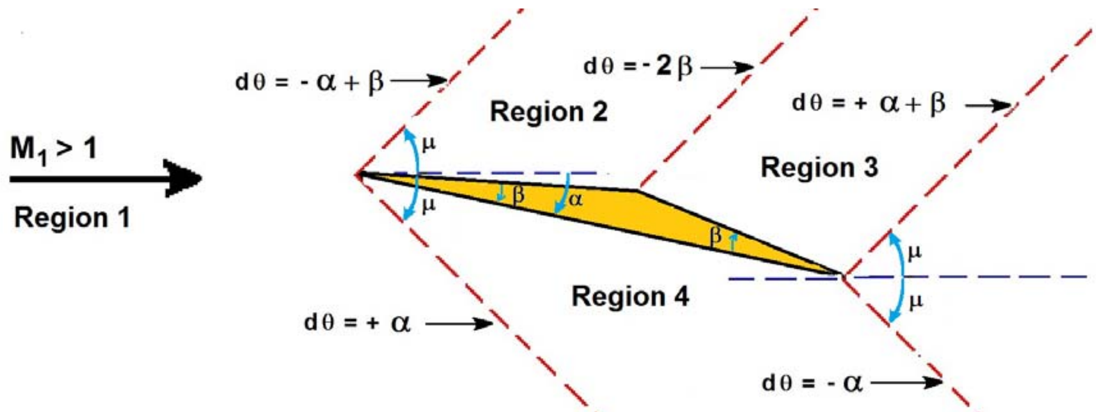

Figure 4: Triangular airfoil at a small angle of attack.

$$p_2 = p_1 + \frac{\gamma M^2(-\alpha+\beta)}{(M^2-1)^{\frac{1}{2}}} \quad ; \quad p_3 = p_1 + \frac{\gamma M^2(-\alpha-\beta)}{(M^2-1)^{\frac{1}{2}}} \quad ; \quad p_4 = p_1 + \frac{\gamma M^2\alpha}{(M^2-1)^{\frac{1}{2}}} \qquad \text{(Bok15)}$$

and, therefore, to first order, the lift and drag coefficients for this triangular airfoil are

$$C_L = \frac{2L}{\rho_1 q_1^2 A} = \frac{4\alpha}{(M_1^2-1)^{\frac{1}{2}}} \quad \text{and} \quad C_D = \frac{2D}{\rho_1 q_1^2 A} = \frac{2(2\alpha^2+\beta^2)}{(M_1^2-1)^{\frac{1}{2}}} \qquad \text{(Bok16)}$$

In words, the "thickness" represented by the angle β has increased the drag without changing the lift (to first order) and therefore the thickness has reduced the performance of the airfoil. Of course, in practice, the "thickness" is necessary to provide the structural strength to withstand the lift and drag forces and, consequently, the design necessarily involves an optimization involving both a structural and a fluid mechanical analysis.

Small deflection supersonic flows past other polygonal objects:

From the preceding examples it is clear that solutions for small deflection supersonic flows past any polygonal object are readily obtained by tracing the flow as it proceeds through the Mach waves generated by each vertex of the object and evaluating the changes in the flow velocity, pressure, temperature and density that occur in each wave. In doing so one recognizes the great advantage over subsonic flows in that in supersonic flow the flow downstream of each wave can not effect the flow upstream. Moreover, as long as the deflections are small, there are two further simplifications. First, since the change across a Mach wave is isentropic (or sufficiently close that the discrepancy is negligible to first order in the small changes) the entire flow is isentropic. Second, since the change in the Mach number across each wave is small, the changes in the pressure, temperature and density are linearly proportional to the deflection angles and do not depend on the changes in the Mach number. Therefore, the difference between the pressure (or temperature or density) at any point in the flow and the pressure (or temperature or density) in the flow far upstream depends only on the difference between the flow inclination at that point and the upstream inclination and does *not* depend on the intermediate states or deflections. Thus the pressure (or temperature or density) at any point is readily determined knowing only the flow direction at that point and the upstream pressure and Mach number. We shall see that these properties of flows with small angular deflections are not necessarily retained when the flow deflection angles are large (see the following sections).

Small deflection supersonic flows past curved surfaces:

The properties described above are particularly useful in developing solutions for small deflection supersonic flows past objects with curved surfaces, such as that sketched in Figure 5. Defining an object with curved

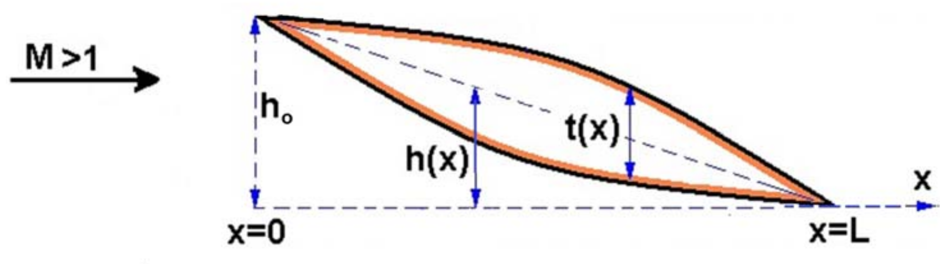

Figure 5: Curved object at small angles to the oncoming supersonic stream.

surfaces described by a mid-thickness profile, $h(x)$, and a thickness, $t(x)$, where x is a coordinate parallel with the oncoming stream as shown in Figure 5, the local flow will be determined by a distribution of Mach waves emanating from the surface and determined by the local change in the surface inclination, $d(h + t/2)/dx$ for the upper surface and $d(h - t/2)/dx$ for the lower surface. (Note that the flow will not consist of discrete Mach waves but a continuous distribution though it may be useful to visualize that distribution as discrete.) Consequently, using equation (Bok8), the difference between the local pressure on the upper surface, $p_1(x)$, and the upstream pressure, p_0, is given by

$$p_1(x) - p_0 \;=\; \frac{\gamma M^2}{(M^2-1)^{\frac{1}{2}}} \frac{d(h+t/2)}{dx} \qquad \text{(Bok17)}$$

and the corresponding pressure distribution, $p_2(x)$, on the lower surface will be given by

$$p_2(x) - p_0 \;=\; -\frac{\gamma M^2}{(M^2-1)^{\frac{1}{2}}} \frac{d(h-t/2)}{dx} \qquad \text{(Bok18)}$$

so that the pressure difference acting on the object in the direction normal to the oncoming flow is

$$p_2(x) - p_1(x) \;=\; -\frac{2\gamma M^2}{(M^2-1)^{\frac{1}{2}}} \frac{dh}{dx} \qquad \text{(Bok19)}$$

The lift and drag follow by integration. For example the lift coefficient, C_L becomes

$$C_L \;=\; \frac{4}{(M^2-1)^{\frac{1}{2}}} \frac{h_0}{L} \qquad \text{(Bok20)}$$

Reflection of a Mach wave:

This example is intended to illustrate two phenomena. First focus on the features associated with the reflection of a Mach wave from a solid surface. In Figure 6 the Mach wave from the leading edge of the airfoil impacts the ground at the Mach angle, μ. The flow upstream of this Mach wave is parallel with the wall and is deflected by the airfoil so that the flow downstream in region2 is parallel with the airfoil and at an angle, α, to the ground. But that inclination is incompatible with the inclination of the ground and so a "reflected" Mach wave must intervene to turn the flow back to parallel with the ground in region 3. Thus the angle of reflection of a Mach wave is equal to the angle of incidence and this will be true of all Mach wave/wall interactions.

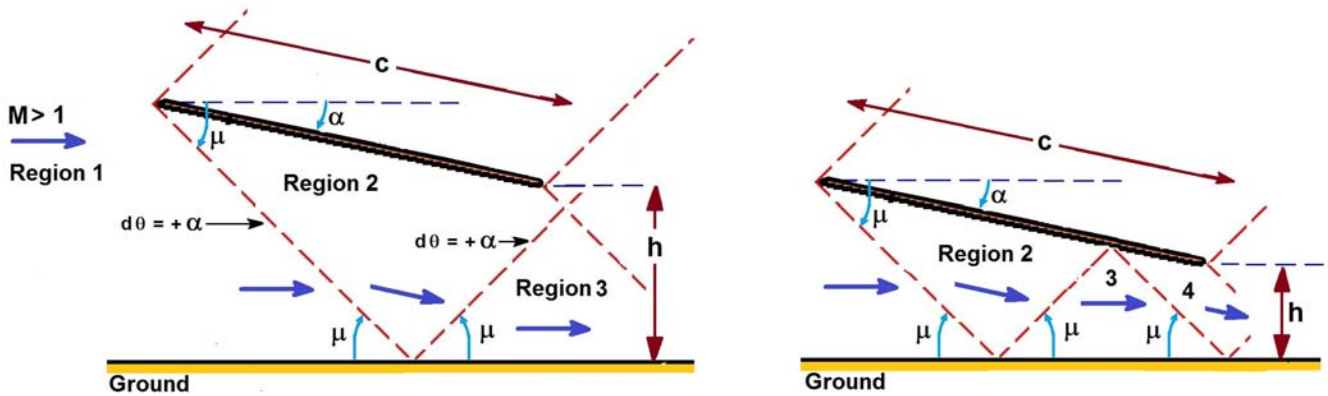

Figure 6: Mach wave reflection and ground effect in supersonic flow. Left: no ground effect. Right: first level ground effect.

The second important feature of this example is the effect on the lift on the airfoil. If the reflected Mach wave does not impact the airfoil as in the left-hand version of Figure 6 then the lift is completely unaffected by the presence of the ground. This will be the case as long as $h/c > 1/M$ or approximately so since the angle α was deemed to be very small. However, if $h/c < 1/M$ (the right-hand version in Figure 6) the reflected shock will strike the pressure surface of the airfoil and affect the lift. Note that the pressure increases as the flow proceeds through the original Mach wave coming from the leading edge of the foil *and* as it proceeds through the reflected wave and so the pressure on the undersurface of the foil downstream of the point of impingement of the reflected wave will be greater than in the left-hand version in Figure 6. Therefore the lift on the foil will be increased and that "ground-effect" will increase as h/c is decreased. Yet another level of interaction with the ground will occur when h/c is decreased to the point at which a second reflection from the ground leads to a second impingement of a Mach wave with the undersurface of the foil but there is little additional benefit to detailing that development here.