Thicknesses and Surface Stress

Contrary to the impression given by the sketches and some of the wording, a laminar boundary has no "edge"; rather, as demonstrated by the Blasius and Falker-Skan solutions, the velocity, u, asymptotes to the external velocity, U. Consequently, we need further deliberation and definition on measures of the "thickness" of a laminar boundary layer; there are three such measures each deserving definition and discussion.

The first measure of the thickness, denoted here by $\delta_{0.99}$, is the n coordinate at which the velocity u has attained a value of $0.99U$. Though this measure is rather arbitrary in comparison with the other two measures, it is, perhaps, the most obvious. As an example we can observe from Figure 2 of section (Bjd) that in the Blasius boundary layer $u/U = dF/d\eta$ reaches a value of 0.99 when $\eta = 2.45$ and therefore

$$\delta_{0.99} = 4.9\left(\frac{\nu s}{U}\right)^{\frac{1}{2}} \quad \text{and} \quad \frac{\delta_{0.99}}{s} = 4.9\left(\frac{\nu}{sU}\right)^{\frac{1}{2}} = 4.9\,(Re_s)^{-\frac{1}{2}} \tag{Bjf1}$$

where Re_s is the Reynolds number based on U and s. Note that $\delta_{0.99}$ has precisely the functional form and dependence on $s^{\frac{1}{2}}$ that was predicted heuristically for the "thickness", δ, in section (Bja).

The second measure is called the *displacement* thickness, δ_D, and is the distance that the external flow has been displaced outwards by the slowing down of the fluid in the boundary layer. Consider the volume flow rate of fluid (per unit depth normal to the plane of the flow) flowing between the solid surface and a fixed location that is a distance, $n = h$, from the solid surface. That volume flow rate is

$$\int_0^h u\,dn \tag{Bjf2}$$

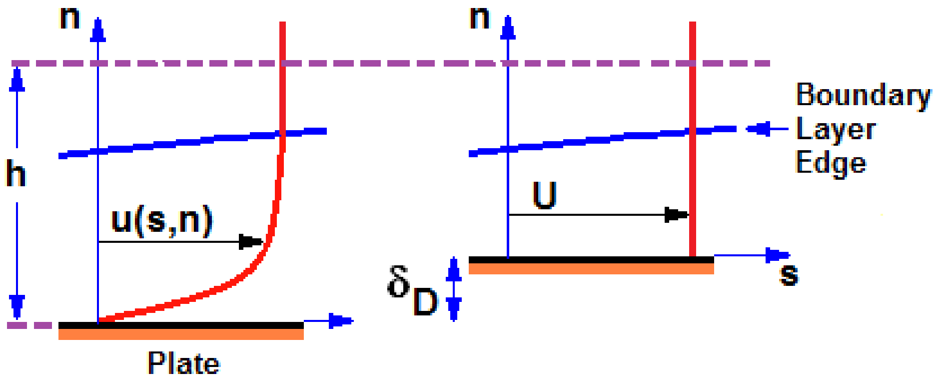

Figure 1: Sketch of the actual velocity profile (left) and hypothetical profile (left) with the same volume flow rate used in defining the displacement thickness, δ_D.

But, in the hypothetical absence of the boundary layer, all the fluid inside $n = h$ would be traveling at U and the same volume flow rate would be contained between $n = \delta_D$ and $n = h$ where δ_D is the distance which the external flow has been displaced outward by the boundary layer. If follows that

$$\int_0^h u\,dn = (h - \delta_D)U \tag{Bjf3}$$

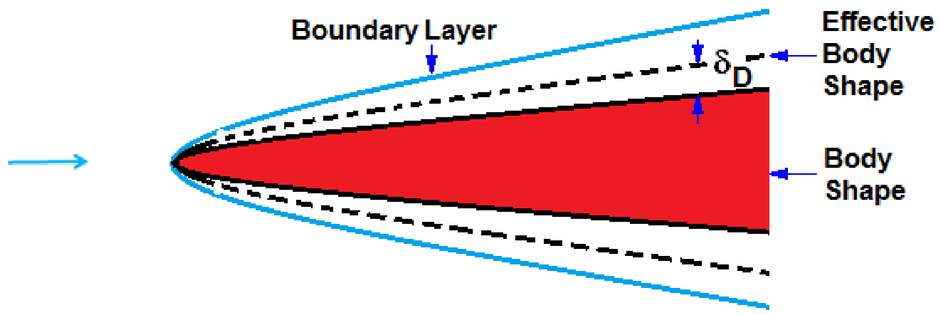

Figure 2: Effect of the displacement thickness, δ_D, on the exterior flow and the effective body shape for that exterior flow.

and therefore

$$\delta_D = \int_0^h 1 - \frac{u}{U}\, dn \tag{Bjf4}$$

and since h is taken to be large and $u \approx U$ for $n > h$ the displacement thickness is defined by

$$\delta_D = \int_0^\infty 1 - \frac{u}{U}\, dn \tag{Bjf5}$$

The displacement thickness represents the effect that the boundary layer flow has on the flow outside the boundary layer. As sketched in Figure 2 the exterior flow essentially encounters an "effective" body which is slightly larger than the actual body by the amount, δ_D. Consequently, in order to improve the accuracy of the solution to the entire flow we should first solve the exterior flow ignoring the boundary layer in order to obtain the first iteration on $U(s)$. Then the boundary layer flow should be solved to find $\delta_D(s)$. Then an effective body shape should be determined and the exterior flow solved gain for that effective shape. Thus an iterative process is identified in which these steps are repeated to convergence. Whether or not convergence can be achieved in a particular flow is an issue that we comment on in later sections.

Note that the Blasius boundary layer has a displacement thickness given by

$$\delta_D \;=\; \int_0^\infty 1 - \frac{u}{U}\, dn \;=\; \left(\frac{4\nu s}{U}\right)^{\frac{1}{2}} \int_0^\infty 1 - \left(\frac{dF}{d\eta}\right) d\eta \;=\; 1.72 \left(\frac{\nu s}{U}\right)^{\frac{1}{2}} \tag{Bjf6}$$

and so the displacement thickness is roughly a third the size of $\delta_{0.99}$ but the functional dependence on Re_s is identical. The Falkner-Skan solutions have a displacement thickness given by

$$\delta_D \;\propto\; \left(\frac{4\nu s}{U}\right)^{\frac{1}{2}} \;\propto\; \left(\frac{4\nu}{C}\right)^{\frac{1}{2}} s^{(1-m)/2} \tag{Bjf6a}$$

where the factor of proportionality is a function of m.

The third thickness, denoted by δ_M, is called the *momentum* thickness and, as we shall see, is related to the shear stress acting at the solid surface and therefore to the viscous drag on the surface. The momentum thickness is defined by

$$\delta_M \;=\; \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dn \tag{Bjf7}$$

In the case of the Blasius boundary layer this yields

$$\delta_M \;=\; \left(\frac{4\nu s}{U}\right)^{\frac{1}{2}} \int_0^\infty \frac{dF}{d\eta}\left(1 - \frac{dF}{d\eta}\right) d\eta \;=\; 0.664 \left(\frac{\nu s}{U}\right)^{\frac{1}{2}} \tag{Bjf8}$$

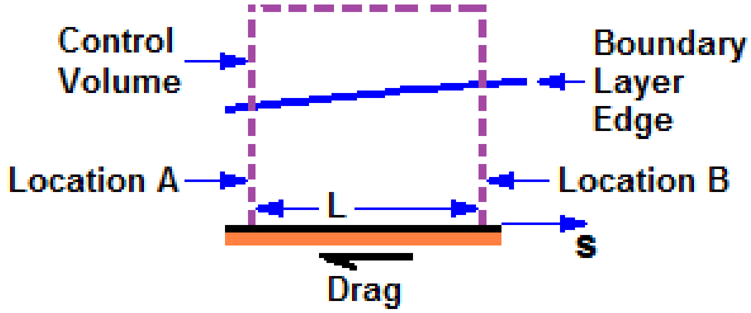

Figure 3: Control volume for Blasius boundary layer.

Consequently, the momentum thickness for the Blasius boundary layer is less than half the displacement thickness and only just over a tenth of $\delta_{0.99}$. The Falkner-Skan momentum thicknesses have the same functional dependence as the displacement thicknesses though the factor of proportionality that depends on m is different.

It is instructive to demonstrate the relation between the momentum thickness and the surface shear shear by analyzing a boundary layer for which dU/ds is zero. Consider the small control volume depicted in Figure 3 consisting of a length, L, of the boundary layer between two locations labelled A and B and extending out into the exterior flow a distance, h, from the surface.

Applying the momentum theorem to this control volume, it follows that the drag force in the $-s$ direction (per unit breadth normal to the plane of the flow) that the wall applies to the fluid in the control volume will be given by

$$\text{Drag} = \rho U^2 \left[\left\{ \int_0^h \frac{u}{U}\left(1-\frac{u}{U}\right)\, dn \right\}_B - \left\{ \int_0^h \frac{u}{U}\left(1-\frac{u}{U}\right)\, dn \right\}_A \right] \qquad \text{(Bjf9)}$$

since the pressure is the same on all sides. Moreover h can be removed to infinity since beyond the boundary layer $u/U = 1$ and hence

$$\text{Drag} = \rho U^2 \left[\left\{ \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)\, dn \right\}_B - \left\{ \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)\, dn \right\}_A \right] = \rho U^2 \left[(\delta_M)_B - (\delta_M)_A \right] \qquad \text{(Bjf10)}$$

Thus the difference between the momentum thickness at two locations is directly proportional to the drag acting on the fluid between those two locations. In fact, this result is just a particular version of the *Karman momentum integral equation* and we delay further discussion until that equation is derived and discussed (section (Bjh)).

However we should make note of a couple of useful results that follow from equations (Bjf10) and (Bjf8). First if we reduce the length, L, to an elemental, ds, then it follows that the drag is equal to $\tau_W ds$ where τ_W is the surface shear stress and therefore

$$\tau_W = \rho U^2 \left\{ \frac{d\delta_M}{ds} \right\} = 0.332 \rho U^{\frac{3}{2}} \left(\frac{nu}{s} \right)^{\frac{1}{2}} \qquad \text{(Bjf10)}$$

and it follows that a friction coefficient defined as $2\tau_W/\rho U^2$ is given by

$$\frac{2\tau_W}{\rho U^2} = \frac{0.664}{(Re_s)^{\frac{1}{2}}} \qquad \text{(Bjf11)}$$

Second, integration to find the total drag, D, on one side of a flat plate of length, L, yields

$$D = 0.664\rho\nu^{\frac{1}{2}}U^{\frac{3}{2}}L^{\frac{1}{2}} \tag{Bjf12}$$

and the corresponding drag coefficient, C_D, becomes

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \frac{1.328}{(Re_L)^{\frac{1}{2}}} \tag{Bjf13}$$

where Re_L is the Reynolds number based on U and the length of the plate, L. Extensive use will be made of these estimates of the viscous drag when we discuss the forces acting on objects in a flow.

The reader may have noticed that, in all of the examples quoted above, we have utilized the Blasius boundary layer for the case in which the pressure gradient and dU/ds are zero. To complete this section it is therefore appropriate to quote some examples in which the external flow is accelerating or decelerating. For this purpose we will utilize the Falkner-Skan solutions for $m > 0$ and $m < 0$ respectively. First we note that the form of the similarity parameter means that all of the three thicknesses, $\delta = \delta_{0.99},\ \delta_D,\ \delta_M$ vary with m according to

$$\delta \propto \frac{\nu^{1/2}s^{(1-m)/2}}{C^{1/2}} \tag{Bjf14}$$

where the factor of proportionality depends upon m and can be deduced from Figure 2 of section (Bje) as needed. Thus the boundary layer thickness in accelerating flows ($m > 0$) increases more slowly than in the Blasius solution; it is being stretched out by the acceleration. Indeed for $m > 1$ the thickness decreases with s. As a first numerical example we quote the result for a 45° half-angle wedge for which $m = 1/3$ and from Figure 2 of section (Bje) u/U becomes 0.99 when $\eta = 1.83$ and therefore

$$\delta_{0.99} = 3.674\ \frac{\nu^{1/2}s^{1/3}}{C^{1/2}} \tag{Bjf15}$$

Another notable example is the behavior of the boundary layer emanating from the stagnation point on a bluff body ($m = 1$); in this case the thickness remains at the constant value of about $2.3(\nu/C)^{\frac{1}{2}}$ until the acceleration begins to decrease.

On the other hand, the boundary layer thickness in a decelerating flow ($m < 0$) increases more rapidly than in the Blasius solution; it piles up like a wave on a beach. For example when $m = -0.05$, $\delta \propto s^{0.525}$ and the thickness increases faster than $s^{0.5}$.