Karman Momentum Integral Equation

Applying the basic integral conservation principles of mass and momentum to a length of boundary layer, ds, yields the *Karman momentum integral equation* that will prove very useful in quantifying the evolution of a steady, planar boundary layer, whether laminar or turbulent. We consider the control volume, $ABCD$, sketched in Figure 1 and bounded by two boundaries normal to the solid surface that are ds apart, by the edge of the boundary layer and by the solid surface. Initially we will presume that the boundary layer has a finite thickness, $\delta(s)$, but it will be seen that the final result does not contain δ and this is therefore a device for the purposes of this derivation.

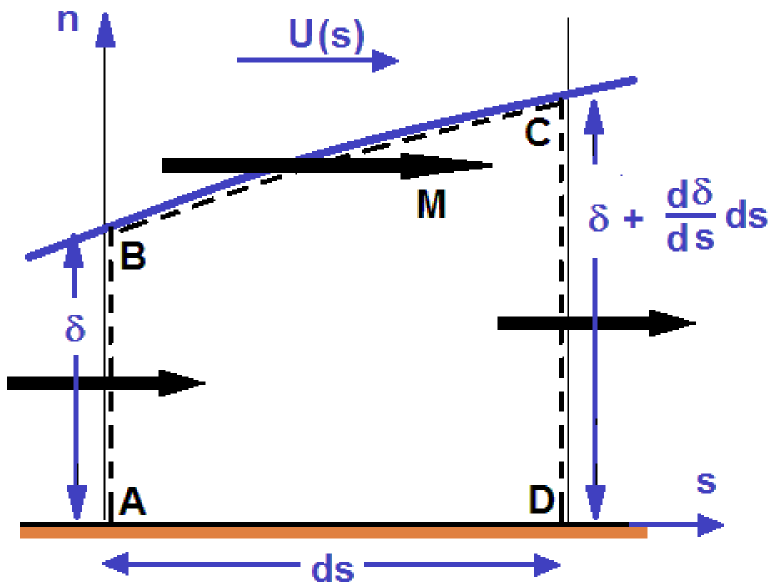

Figure 1: Control volume, $ABCD$, consisting of a length of boundary layer, ds.

First we identify the mass flow rates involved in the steady, planar flow through the boundaries of this control volume where, throughout this derivation the control volume (CV) is assumed to have unit depth normal to the plane of the flow. The mass flow rate into the control volume through AB will be

$$\int_0^\delta \rho u \; dn \tag{Bjh1}$$

and the flow rate out through CD will be

$$\int_0^\delta \rho u \; dn + ds \, \frac{d}{ds} \left\{ \int_0^\delta \rho u \; dn \right\} \tag{Bjh2}$$

Since there is no mass flow through the boundary DA it follows from conservation of mass that the mass flow rate, M, into the control volume through the boundary BC must be

$$M \;=\; ds \, \frac{d}{ds} \left\{ \int_0^\delta \rho u \; dn \right\} \tag{Bjh3}$$

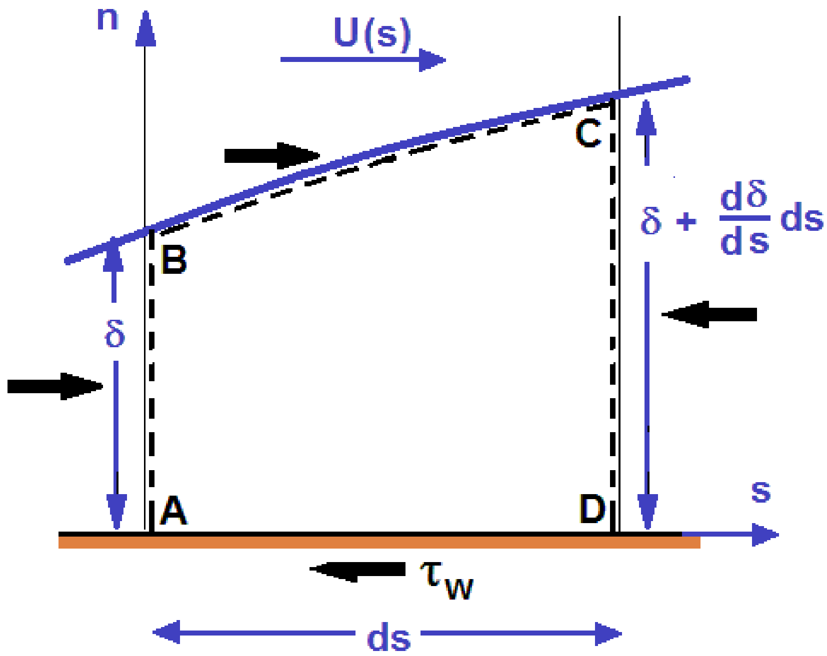

Figure 2: Forces on the control volume consisting of a length of boundary layer, ds.

Second, we identify the momentum fluxes (in the s direction) *out of the control volume* through each of the boundaries. The momentum flux out of the control volume through AB is

$$-\int_0^\delta \rho u^2 \, dn \tag{Bjh4}$$

and momentum flux out of the control volume through CD is

$$\int_0^\delta \rho u^2 \, dn + ds \, \frac{d}{ds}\left\{\int_0^\delta \rho u^2 \, dn\right\} \tag{Bjh5}$$

Moreover, the momentum flux out of the control volume through BC is

$$-MU \;=\; Uds \, \frac{d}{ds}\left\{\int_0^\delta \rho u \, dn\right\} \tag{Bjh6}$$

and therefore the net momentum flux out of the control volume is

$$ds \, \frac{d}{ds}\left\{\int_0^\delta \rho u^2 \, dn\right\} \;-\; Uds \, \frac{d}{ds}\left\{\int_0^\delta \rho u \, dn\right\} \tag{Bjh7}$$

We now utilize the linear momentum thereom for steady, planar flow to equate this net momentum flux out of the control volume to the net force acting on the control volume in the s direction. Utilizing the boundary layer result of section (Bjb) that $\partial p/\partial n = 0$, the forces on the sides AB and CD in the s direction are respectively

$$p\delta \qquad \text{and} \qquad -p\delta - ds \, \frac{\partial(p\delta)}{\partial s} \tag{Bjh8}$$

while the force on the top-side in the s direction is

$$(p + \Delta p)ds \, \frac{d\delta}{ds} \tag{Bjh9}$$

where $(p + \Delta p)$ is some effective pressure acting on BC intermediate between p and $[p + ds(dp/ds)]$. Combining the forces in the expressions (Bjh8) and (Bjh9) and neglecting the term involving Δp since it is second order, it transpires that the net force acting on the control volume in the s direction is

$$-\delta \, \frac{dp}{ds} \, ds - \tau_W \, ds \tag{Bjh10}$$

where τ_W is the shear stress that the solid surface imposes on the fluid in the control volume in the $-s$ direction. Consequently the linear momentum thereom yields

$$\delta \, \frac{dp}{ds} \, ds + \tau_W \, ds \;=\; ds \, \frac{d}{ds} \left\{ \int_0^\delta \rho u^2 \, dn \right\} \;-\; U ds \, \frac{d}{ds} \left\{ \int_0^\delta \rho u \, dn \right\} \tag{Bjh11}$$

Cancelling through ds and substituting $dp/ds = -\rho U \; dU/ds$ from Bernoulli's equation, equation (Bjh11) can be written as

$$\frac{\tau_W}{\rho} \;=\; \frac{d}{ds} \left\{ U^2 \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) \, dn \right\} + U \frac{dU}{ds} \left\{ \int_0^\delta \left(1 - \frac{u}{U}\right) \, dn \right\} \tag{Bjh12}$$

Noticing that both the integrands in the above expression become zero when $n > \delta$, the limits in the two integrals can be extended from δ to ∞ without changing their values and, using the definitions of the displacement thickness, δ_D, and the momentum thickness, δ_M, (see equations (Bjf5) and (Bjf7)) equation (Bjh12) becomes

$$\frac{\tau_W}{\rho} \;=\; \frac{d}{ds} \left\{ U^2 \delta_M \right\} + \delta_D U \frac{dU}{ds} \tag{Bjh12}$$

This is the *Karman momentum integral equation* that, given $U(s)$, provides a functional relation between $\tau_W(s)$, $\delta_D(s)$ and $\delta_M(s)$ for steady, planar boundary layer flow. Notice that it does not explicitly involve δ and hence that quantity does not need explicit definition. Notice also that in the Blasius case, dU/ds, the Karman momentum integral equation reduces to the previously-derived equation (Bjf10).

The Karman momentum integral equation provides the basic tool used in constructing approximate solutions to the boundary layer equations for steady, planar flow as will be further explored in section (Bji). Moreover, nowhere in the derivation was it necessary to assume that the boundary layer flow was laminar and hence the Karman momentum integral equation can also be used to investigate the behavior of turbulent boundary layers as will be pursued in section (Bjj).