Falkner-Skan Solutions to Laminar Boundary Layer Equations

Falkner and Skan supplemented the exact solutions to steady, planar, laminar boundary layer flows by obtaining a family of solutions for external velocities, $U(s)$, that have the simple polynomial form $C\ s^m$ where C and m are constants. With this input the boundary layer equations (Bjb20) and (Bjb21) become

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \tag{Bje1}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = mC^2\ s^{2m-1} + \nu\frac{\partial^2 u}{\partial n^2} \tag{Bje2}$$

and, using the stream function, ψ, given by

$$u = \frac{\partial \psi}{\partial n} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial s} \tag{Bje3}$$

the single governing equation can be written as

$$\frac{\partial \psi}{\partial n}\frac{\partial^2 \psi}{\partial s \partial n} - \frac{\partial \psi}{\partial s}\frac{\partial^2 \psi}{\partial n^2} = mC^2\ s^{2m-1} + \nu\frac{\partial^3 \psi}{\partial n^3} \tag{Bje4}$$

The boundary conditions are

1. $\psi = 0$ on the solid surface, $n = 0$, since v=0
2. $\partial\psi/\partial n = 0$ on the solid surface, $n = 0$ since u=0
3. $\partial\psi/\partial n \rightarrow U$ as $n \rightarrow \infty$ since $u \rightarrow U$

We recognize that the Blasius solution is one of the members of this family, namely the one with $m = 0$.

Before proceeding with the development of the similarity solution that Falkner and Skan obtained, it is valuable to pause and identify the physical or geometric contexts in which such external velocity distributions might be relevant. Toward this end, we seek irrotational flows in which the velocity along the solid surface behaves like $U = C\ s^m$. There are two physical circumstances for which this can be envisaged. The first is a precise global view of the potential flow past a wedge in a uniform stream; the second is an approximate, local application in which we seek to simulate the local behavior of the boundary layer in a localized region where the velocity U and the acceleration (or deceleration) are used to determine C and m.

Consider first the precise global interpretation involving the flow past a symmetric wedge as depicted in Figure 1. It transpires that this steady, planar, potential flow has a surface velocity, U which varies like $U = C\ s^m$, with distance s measured from the vertex or stagnation point of the flow. Moreover, the half-angle of the wedge, $\beta\pi/2$, determines m according to

$$\beta\ =\ \frac{2m}{m+1} \tag{Bje5}$$

Consequently half-angles of 45°, 60° and 90° correspond to m values of 1/3, 1/2 and 1 respectively; for example, the boundary layer emanating from the stagnation point on a bluff body ($\alpha = 90°$) would be locally governed by $m = 1$.

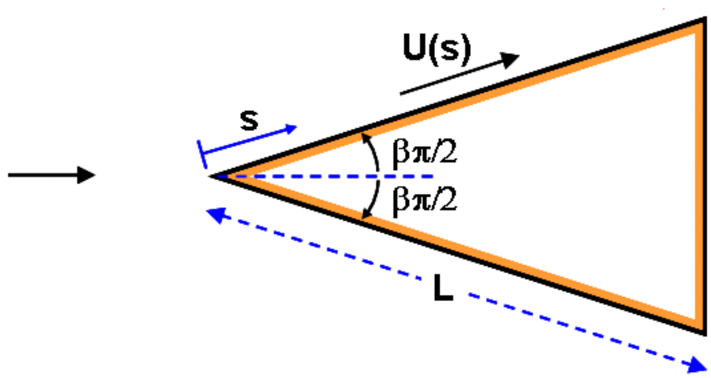

Figure 1: Flow of a uniform stream past a wedge.

The second interpretation would be to evaluate m using a known surface velocity distribution, $U(s)$, to obtain locally pertinent m values according to

$$m = \frac{d(\ln U)}{d(\ln s)} \tag{Bje6}$$

and then to apply the Falkner-Skan solution for that m to determine the approximate local development of the boundary layer. This approach is described further in section (Bji). We note that $m < 0$ corresponds to a locally decelerating external flow corresponding to an "adverse pressure gradient" whereas $m > 0$ corresponds to a locally accelerating external flow corresponding to a "favorable pressure gradient". For further discussion on the effects of an accelerating or decelerating external flow the reader is referred to section (Bjg).

The first step in seeking a solution to the governing equation (Bje4) is to determine the appropriate similarity variable, η. While it is possible to begin with a general form for the similarity variable like $\eta = s^\xi\, n^\zeta$ and to determine the unknowns ξ and ζ that allow a similarity solution, we will, in the interests of brevity bypass that part of the solution by anticipating that the similarity variable that works is the same as for the Blasius solution except that U is now a function of s. In other words

$$\eta = n\left(\frac{U}{4\nu s}\right)^{\frac{1}{2}} = n\left(\frac{Cs^{m-1}}{4\nu}\right)^{\frac{1}{2}} \tag{Bje7}$$

and the solution is similarly given by the streamfunction, $\psi(s, n)$, where

$$\psi = (4\nu U s)^{\frac{1}{2}} F(\eta) = (4\nu C s^{m+1})^{\frac{1}{2}} F(\eta) \qquad \text{and} \qquad \frac{u}{U} = \frac{dF}{d\eta} \tag{Bje8}$$

where, $F(\eta)$, may be different than in the Blasius solution. By utlizing the relations (Bje7) and (Bje8) to evaluate the derivatives of ψ the governing equation (Bje4) becomes

$$2(m+1)F\frac{d^2F}{d\eta^2} + \frac{d^3F}{d\eta^3} + 4m - 4m\left(\frac{dF}{d\eta}\right)^2 = 0 \tag{Bje9}$$

The boundary conditions on $F(\eta)$ remain the same as in the Blasius case namely

1. $v = 0$ on the solid surface, $\eta = 0$, and therefore $(dF/d\eta)_{\eta=0} = 0$
2. $u = 0$ on the solid surface, $\eta = 0$, and therefore $F_{\eta=0} = 0$

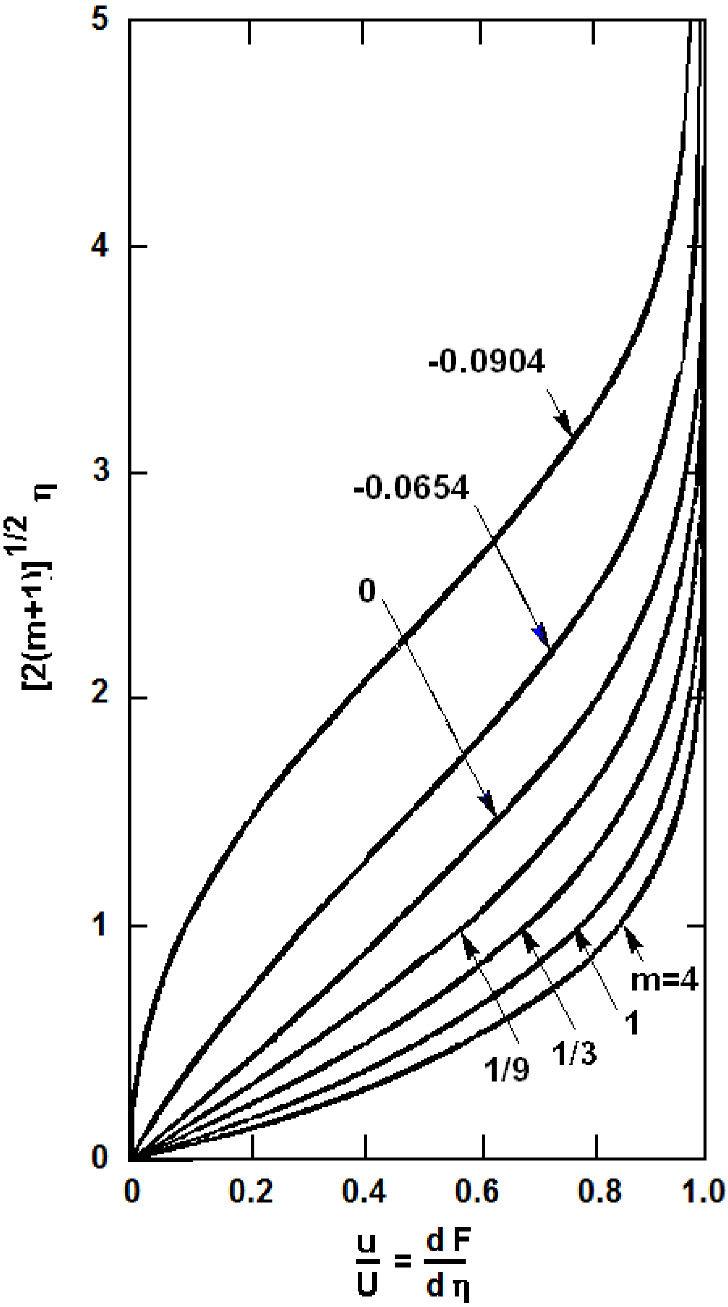

Figure 2: Velocity profile shapes from Falkner-Skan solutions for various values of m.

3. $u \rightarrow U$ as $\eta \rightarrow \infty$ and therefore $(dF/d\eta)_{\eta\rightarrow\infty} \rightarrow 1$

Moreover, as in the Blasius case, there are no parameters in this governing equation other than m and the family of solutions need only be generated once. The family of solutions for $dF/d\eta$ is presented in Figure 2 for a number of values of the paramter, m.

Notice how the shape of the velocity profile changes with m. The shape for $m = 0$ is, of course, identical to the shape from the Blasius solution with no acceleration. However as m is increased and the external flow acceleration increases, the shape becomes significantly blunter and the gradient of the velocity at the solid surface, $\partial u/\partial n$, increases. On the other hand as m is decreased to negative values corresponding to a decelerating external flow, the velocity profile becomes less blunt and, in succession, reaches two important shapes. The first occurs at $m = -0.654$ when a point of inflection first occurs in the velocity profile. This is significant because, as discussed later in section (Bk), the boundary layer flow becomes unstable when it features a point of inflection. The second critical development occurs at $m = -0.904$; at this value the velocity gradient, $\partial u/\partial n$, at the solid surface becomes zero. This is significant for two coupled reasons. First this means that the shear stress at the solid surface has become zero. Moreover, if m is reduced below -0.904, $\partial u/\partial n$ becomes negative and the velocity profile develops a layer of reverse flow next to the solid surface. Both of these important developments in the shape of the laminar velocity profile will be further discussed in sections that follow.