Approximate Methods

Having derived the Karman momentum integral equation (KMIE),

$$\frac{\tau_W}{\rho} \;=\; \frac{d}{ds}\left\{U^2\delta_M\right\} + \delta_D U \frac{dU}{ds} \tag{Bji1}$$

we are now in a position to add to the Blasius and Falkner-Skan solutions by developing approximate methods for solving the laminar boundary layer equations. These methods utilize approximations to relate the three unknown functions in the KMIE, namely $\delta_D(s)$, $\delta_M(s)$ and $\tau_W(s)$ and thus reduce the number of unknown function from three to one. One way in which this has been done to do this is rewrite equation (Bji1) as

$$\frac{\tau_W}{\rho U^2} \;=\; \frac{d\delta_M}{ds} + \left\{2 + \frac{\delta_D}{\delta_M}\right\}\frac{\delta_M}{U}\frac{dU}{ds} \tag{Bji2}$$

where the left-hand side is two times the usual skin friction coefficient, c_f. Then, to form a differential equation for $\delta_M(s)$ we define two *profile parameters* traditionally denoted by T and H and defined by

$$T \;=\; \frac{\tau_W\delta_M}{\mu U} \;=\; \frac{\delta_M}{U}\left(\frac{\partial u}{\partial n}\right)_{n=0} \;=\; \left\{\frac{\partial(u/U)}{\partial(n/\delta_M)}\right\}_{n=0} \tag{Bji3}$$

and

$$H \;=\; \frac{\delta_D}{\delta_M} \;=\; \frac{\left[\int_0^\infty \left(1-\frac{u}{U}\right) d\left(\frac{n}{\delta_M}\right)\right]}{\left[\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) d\left(\frac{n}{\delta_M}\right)\right]} \tag{Bji4}$$

The relation of T and H to the velocity profile is shown schematically in Figure 1. It follows that the equation (Bji2) can be written as

$$\left(\frac{\nu T}{U}\right)\frac{1}{\delta_M} \;=\; \frac{d\delta_M}{ds} + (2+H)\frac{\delta_M}{U}\frac{dU}{ds} \tag{Bji5}$$

If T and H were known, this equation could be solved for $\delta_M(s)$, given $U(s)$. Note that T and H are functions only of the shape of the velocity profile (see Figure 1) and therefore of the dimensionless velocity profile

$$\frac{u(n/\delta_M)}{U} \tag{Bji6}$$

though they will also normally be functions of s.

The approximate methods described here are based on the judgment that the results are not very sensitive to the precise shape of the velocity profile as long as that profile satisfies the basic boundary conditions which are usually:

- $u = 0$ at $n = 0$
- $u \rightarrow U$ as $n \rightarrow \infty$

though variations occur, for example if the solid surface is moving or is porous. Sometimes simple approximate velocity profile shapes are assumed, for example, one of the Blasius or Falkner-Skan profiles. Once the profile shape has been assumed and the form of the function (Bji6) is known, numerical values of T and H follow from the definitions (Bji3) and (Bji4). For example, for the Blasius profile $T = 0.221$ and

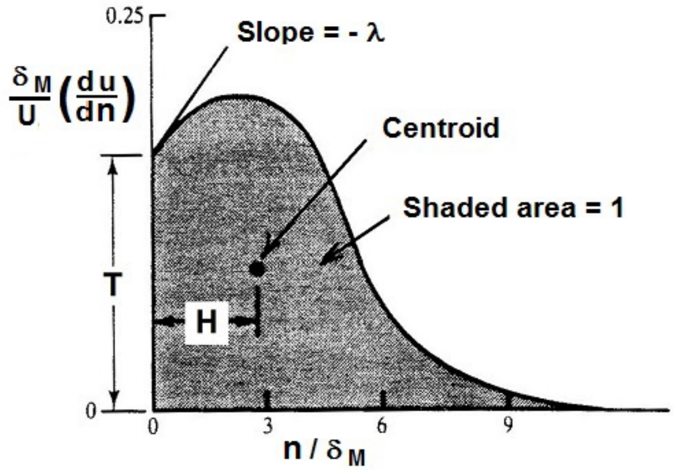

Figure 1: Relation between T, H and λ and the boundary layer profile.

$H = 2.59$. Therefore, if the velocity profile shape was the same for all s (as it would be for the Blasius and Falkner-Skan flows), equation (Bji5) would then be fully defined and we could proceed to a solution for those known and fixed values of T and H. However, even this objective is not readily achieved analytically except when U is some simple function of s.

The next step is to decide on a family of approximate velocity profile shapes that will cover the variety of shapes that occur with accelerating and decelerating external flows. A variety of parametric shapes have been suggested, some governed by more than one parameter. However, since most of those more complex methods have been made obsolete by the availability of computational fluid dynamics methods for the solution not only of boundary layer flows but also for the external flow, we will confine the present discussion to velocity profile families governed by a single parameter. The best known of these is the method presented by Thwaites (1960) who observed that T and H are close to being functions only of the variable, λ, where

$$\lambda = \frac{\delta_M^2}{\nu}\frac{dU}{ds} \tag{Bji7}$$

where Thwaites obtained the functions $T(\lambda)$ and $H(\lambda)$ shown in Figure 2 from a combination of experimental data and the Blasius and Falkner-Skan solutions. Note that λ can be related to the Falkner-Skan index m by observing that, in the Falkner-Skan solution,

$$U \;=\; Cs^m \qquad \text{and} \qquad \delta_M^2 \;=\; \frac{\nu}{Cs^{m-1}} f(m) \tag{Bji8}$$

where $f(m)$ is some function of m and therefore λ is a function only of m and not of s. That functional relation is plotted in Figure 2. Thus, using λ as surrogate for m, Thwaites' approximate method deploys the Falkner-Skan family of velocity profiles. The Karman momentum integral equation (Bji5) is written as

$$\frac{d(\delta_M^2)}{ds} \;=\; \frac{2\nu}{U}\left(T - \lambda(H+2)\right) \tag{Bji9}$$

and Thwaites found that, empirically,

$$T - \lambda(H+2) \;\approx\; 0.225 - 3\lambda \qquad \text{and} \qquad T \;\approx\; (\lambda + 0.09)^{0.62} \tag{Bji10}$$

so that equation (Bji9) could be approximated by

$$\frac{d(\delta_M^2)}{ds} \;=\; \frac{0.45\nu}{U} - \frac{6\delta_M^2}{U}\frac{dU}{ds} \qquad \text{or} \qquad \frac{d}{ds}\left\{\delta_M^2 U^6\right\} \;=\; 0.45\nu \tag{Bji11}$$

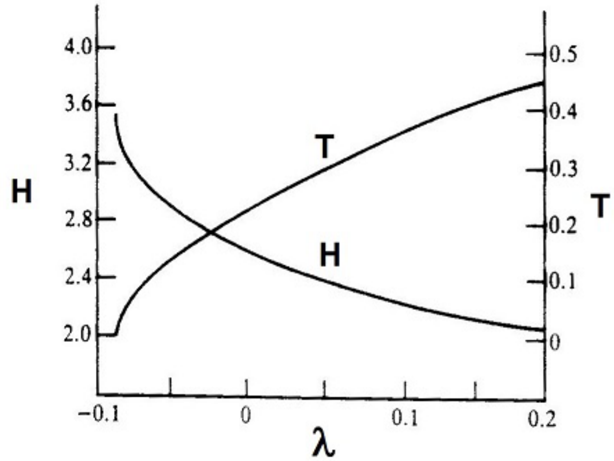

Figure 2: Thwaites relations for $T(\lambda)$ and $H(\lambda)$.

Integration yields the approximate solution for the momentum thickness

$$\delta_M^2 = \left\{\delta_M^2\right\}_{s=s_0} + \frac{0.45\nu}{U^6}\int_{s_0}^{s} U^5 ds \tag{Bji12}$$

so that, given the external velocity, $U(s)$, and the momentum thickness, δ_{s_0} at one location, s_0, the momentum thickness at any downstream location can be estimated. Once $\delta_M(s)$ has been obtained, $\lambda(s)$, $T(s)$ and $H(s)$ follow from the relation (Bji7) and Figure 2. Consequently the other boundary layer properties, $\delta_D(s)$ and $\tau_W(s)$ follow from $T(s)$ and $H(s)$.

At this point some simple examples are is appropriate. The simplest is, of course, the Blasius case in which U is constant and, when $\delta_{s_0} = 0$, it follows from equation (Bji12) that $\delta_M = 0.671(\nu s/U)^{\frac{1}{2}}$ which is satisfactorily close both in numerical value and functional dependence to the actual Blasius solution, $\delta_M = 0.664(\nu s/U)^{\frac{1}{2}}$. This is not, however, surprising since Thwaites used the Blasius solution as one of his benchmarks. Other examples are, of course, the Falkner-Skan flows with $U = Cs^m$ for which equation (Bji12) yields:

$$\delta_M = \left\{\frac{0.45\nu}{C(5m+1)}\right\}^{\frac{1}{2}} s^{(1-m)/2} \tag{Bji13}$$

This has the same functional dependence as indicated in equation (Bjf6a).

It should emphasized again that these approximate methods have been largely replaced by numerical solutions of the laminar boundary layer equations. Perhaps the most valuable usage of the Thwaites method is to provide a straightforward analytical method of estimating the point of separation of a laminar boundary layer. As we have previously documented, the Falkner-Skan profile at separation has an m value of -0.0904. This corresponds to values of $\lambda = -0.09$ ($T = 0$ and $H = 3.5$). Therefore, to make an estimate of the location of laminar boundary layer separation the integration in equation (Bji12) should continue until the s location at which that critical value of λ (or $T = 0$) is encountered.