Boundary Integral Methods

When the fluid flow equations to be solved are linear (for example potential flow) so that solutions can be superimposed in the process of creating the desired solution, then it may be possible to utilize boundary integral methods in order to simulate the required flow. This involves distributing individual singular solutions over the surface of the solid boundaries (and, perhaps, in their interior) and then summing the flows induced by each of the singularities and solving for their strength by applying appropriate boundary conditions. Such a procedures are called **boundary integral methods**. They have the advantage over the finite difference or finite element methods (field methods for short) in that the number of unknowns is proportional to the number of discretized elements on the boundary surface (proportional to N^2, the number of elements on the surface) whereas the typical number of unknowns in the field methods would be N^3.

As an example of the boundary element method we will outline the vortex sheet method for planar

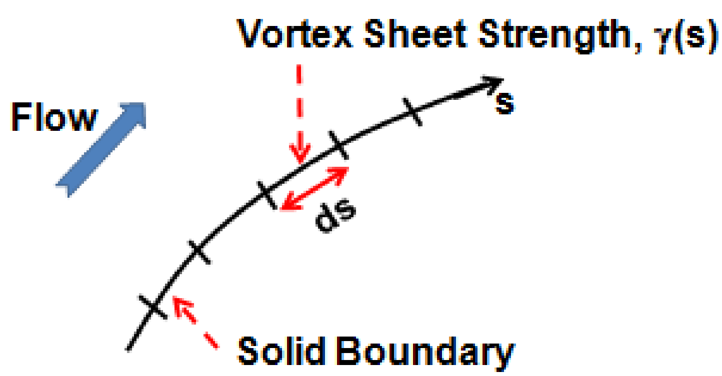

Figure 1: Section of the solid boundary of a planar potential flow.

potential flows also described elsewhere in this text. This is based on the planar potential flow induced at any general point in the flow field, (x, y), by the vortex sheet element of strength, $\gamma(s)$, on the boundary surface of the flow (see Figure 1) whose position is $(x_s(s), y_s(s))$. That potential flow is

$$\phi = \frac{\gamma(s)\ ds}{2\pi} \arctan\left\{\frac{(y - y_s(s))}{(x - x_s(s))}\right\} \quad ; \quad \psi = -\frac{\gamma(s)\ ds}{4\pi} \ln\left\{(x - x_s(s))^2 + (y - y_s(s))^2\right\} \tag{Ob1}$$

according to equation (Idc3). Summing the contributions from all the elements of the surface, S, the flow field is then simulated by

$$\phi = \int_S \frac{\gamma(s)\ ds}{2\pi} \arctan\left\{\frac{(y - y_s(s))}{(x - x_s(s))}\right\} \quad ; \quad \psi = -\int_S \frac{\gamma(s)\ ds}{4\pi} \ln\left\{(x - x_s(s))^2 + (y - y_s(s))^2\right\} \tag{Ob2}$$

plus contributions from other components such as a uniform stream. The velocity components then follow by differentiation of ϕ or ψ. Of course, the vortex element strengths, $\gamma(s)$, are, as yet, unknown and this determination is made by applying the boundary condition that the velocity induced normal to the surface of each element must be zero.

Numerically the method therefore proceeds as follows. The solid surface(s) is divided into J elements of strength, γ_j, $j = 1, ..., J$, length, s_j, and position $(x - x_j), (y - y_j)$. Then using

$$\phi = \sum_{j=1}^{J} \frac{\gamma_j \ s_j}{2\pi} \arctan \left\{ \frac{(y - y_j)}{(x - x_j)} \right\} \quad ; \quad \psi = - \sum_{j=1}^{J} \frac{\gamma_j \ s_j}{4\pi} \ln \left\{ (x - x_j)^2 + (y - y_j)^2 \right\} \tag{Ob3}$$

plus contributions from other components such as a uniform stream, the induced velocities at the center of each of the surface elements are determined (in doing so we omit from the summations any contribution from the vortex element where we are evaluating the velocity). This yields numerical expressions for the J normal velocities in the center of each of the elements. These must all be zero and this generates J equations which must then be solved to determine the J vortex sheet strengths, γ_j. This is the simplest form of the boundary element method based on vortex sheet elements. It has the advantage over the doublet distribution method in that the known shape of the object is inputted at the beginning rather than simply being discovered as the calculation proceeds.

The basic methodology described above was first applied to planar potential flows. Later the method was extended to axisymmetric flows with the development of the velocity potential due to a short cylindrical vortex element. The first work on a three-dimensional non-lifting problem was based on an unknown source distribution over the boundary surface and was commonly referred to as the *panel method*. This was subsequently extended to include lifting surfaces using either vortices or dipoles normal to the surface.