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Shear Flows of Rapidly Flowing Granular Materials

Shear flows of granular materials are studied in an open channel. The wall shear is calculated from an open channel momentum equation which includes the density variations in the flow. An experimental technique was developed that allowed the measurement of the average density of the flow at different longitudinal locations in the channel. Two sizes of glass beads are examined and results show the variations in the wall shear as a function of various dimensionless parameters.

I Introduction

In both industrial and commercial applications, large amounts of material are transported in granular form. This includes the handling of such substances as coal, metal ores, shale, dry chemicals, and grain. In addition, flowing granular streams are being considered for some advanced concepts for solar power plants and fusion reactor chambers. In order to design the equipment for these varied applications in an effective and economical way, it is necessary to obtain a thorough understanding of the flow characteristics of granular materials. These design needs have already motivated extensive analytical and experimental investigations of granular flows. At this time, however, there is still no clear understanding of the constitutive relations that govern the motion of granular materials. The general field is still in a stage of development comparable to that of fluid mechanics before the advent of the Navier-Stokes relations. The present work was designed to contribute information which may prove helpful in the eventual formulation of a constitutive law. The data also provides practical information on the friction due to the flow of granular materials over a smooth wall.

II Review of Related Work

R. A. Bagnold is credited with the development of the modern research in granular material flows with his experiments and theories dating to the early 1950's (Bagnold, 1954, 1956, 1966). Recent progress has been described in an excellent review by Savage (1984). The reader is referred to this work and no attempt will be made here to offer any extensive account of the background material. For our purposes, perhaps the most pertinent recent work is that performed by Savage (1979, 1983), Sayed and Savage (1983), Hanes and Inmann (1985), Craig et al. (1986), and Bailard (1978) all of whom used Couette flow devices or open channels to experimentally study the shear of granular materials. With the aid of Couette flow devices, the behavior of the shear stress as a function of the shear rate, the normal pressure, and the particle size and density was studied. Sayed and Savage (1983), Augenstein and Hogg (1974, 1978), and Bailard (1978) were able to compute velocity and density profiles for the flow along an inclined chute based on certain constitutive assumptions. Also applicable is the work by Campbell et al. (1985a) on granular flow in an inclined chute. This experimental work yielded some preliminary results on the shear stress in open channel flows; however, their study did not account for the density changes of the flow in the channel. The present study shows that these density changes play an important role in adequately describing a granular material flow. The computational work carried out by Campbell (1982), and Campbell and Brennen (1985b), involved a statistical analysis of an assembly of particles flowing down a chute. The results give some indication of the magnitude and distribution of velocity and density, as well as the fluctuational components of the flow field, and may serve as background information for the eventual formulation of the constitutive laws.

III Experimental Installation

The present investigation was designed to obtain further information on the parameters that influence the shear in a flowing granular material. For simplicity of analysis, an open channel was proposed as the test section. With this purpose in mind, a large installation was constructed which would allow for continuous flow in a relatively wide open channel. A wide channel was selected so that the effect of the side walls could be minimized and a continuous operating loop allowed proper adjustment of the flow as well as more accurate measurements of the flow quantities by eliminating the time constraints on

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the measurements. An installation was built consisting of a 3 m long, 150 mm wide channel which could be placed at angles up to 40 deg from the horizontal position. A high-speed mechanical conveyor delivers the material to an upper hopper from which the material flows into the channel. The discharge from the channel is collected in a lower hopper which feeds the material to the conveyor (See Fig. 1).

The measurements taken were the mass flow rate, profiles of the depth of the flowing stream, and the local density. The mass flow rate was determined by monitoring the rate of depletion of the upper feed hopper through a graduated transparent panel. Some confirmation of these flow rates were obtained by collecting and weighing the material discharging from the chute in a given amount of time. The depth of the flow in the channel was measured at several points along the chute by means of point depth probes identical to those commonly used in open-channel hydraulics. A simple yet effective method was developed to measure an average density of the flowing material. A device consisting of two plates connected by a handle was suddenly pushed into the flow thereby trapping the flowing material in the space between the plates. The trapped material was then collected and weighed, and the average density of the original stream was computed from this weight, the measured depth of the original stream, and the dimensions of the trap. For any given flow, this procedure could be repeated at different locations along the channel which allowed the density gradients to be evaluated in the direction of flow. From such density measurements, meaningful average velocities and densities could be computed. The ability to obtain an indication of the average density in this way was an important factor in allowing a more realistic interpretation of the data for open channel granular material flows. A drawing of the channel, point probes, and density device may be seen in Fig. 2.

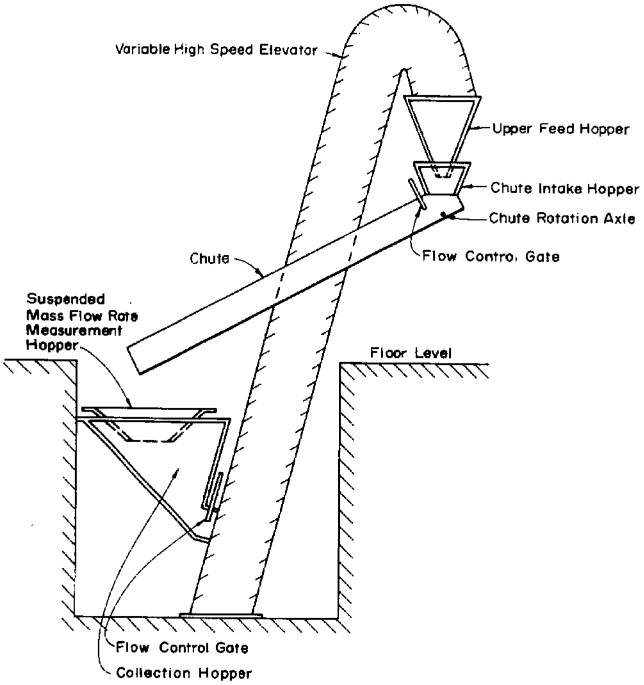

Fig. 1 Schematic of the experimental facility

IV Computation of Shear

In addition to the flow quantities just mentioned, the shear on the bottom of the channel was determined by means of the following considerations. First the momentum equation was written for the flow in the channel. The flow was assumed to be steady and one-dimensional and the pressure distribution was taken to be hydrostatic. It should be noted, however, that the density is a variable and these density changes must be accounted for in the momentum equation. With these assumptions the resulting equation may be written as:

$$\frac{\tau_w P}{\rho_p \nu g h \cos\theta} = \tan\theta + \frac{dh}{dx}(\mathrm{Fr}^2 - 1) + \frac{h}{\nu}\frac{d\nu}{dx}(\mathrm{Fr}^2 - 1/2) \qquad (1)$$

where τ_w represents the shear at the channel bottom, ρ_p the particle density, ν the solid fraction, h the depth of flow, θ the channel inclination, x the distance along the channel, and g is the gravitational acceleration. The Froude number, Fr, was defined as $U/(gh\cos\theta)^{1/2}$, where U is the average velocity. The first term on the right-hand side corresponds to the component of the gravitational force in the direction of flow. The last two terms account for the acceleration of the flow and depend on the gradient of the depth of flow, (dh/dx), and on the gradient of the density, $(d\nu/dx)$. The accuracy of this equation could be improved by the inclusion of profile parameters which depend on the shape of the velocity and density profiles over the depth of flow. However, in the absence of this information, these profile parameters are assumed to be unity. The factor P on the left-hand side accounts for the friction effect of the side walls and may be expressed as

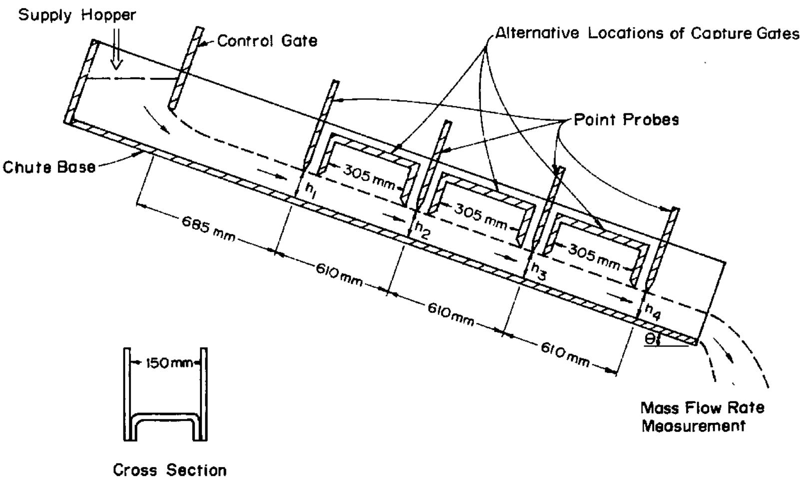

Fig. 2 Schematic of the test channel and density device

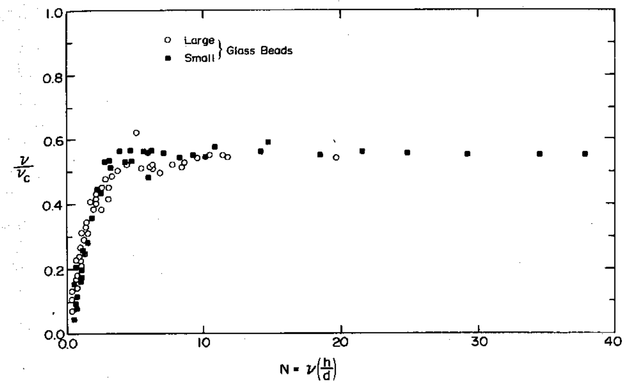

Fig. 3 The variation of the solid fraction, ν, as a function of the characteristic number of particle layers in the flow, N, for both sizes of glass beads

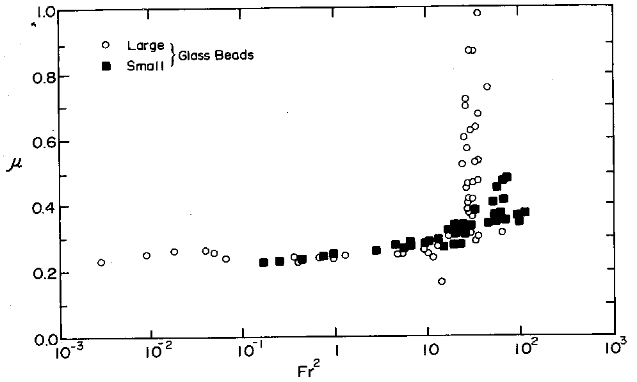

Fig. 4 The variation of the friction coefficient, μ, as a function of Froude Number Fr^2. (Both 3 mm and 0.3 mm glass beads are shown.)

Table 1 Material properties

Average size of small glass beads	0.26 mm
Average size of large glass beads	2.94 mm
Bulk specific gravity (both sizes)	1.5
Wall friction angle against aluminum (both sizes)	15 to 18 deg
Internal friction angle (both sizes)	18 to 26 deg

$$P = (1 + \beta h/b)$$

where b is the width of the channel, h is the flow depth, and β is a constant which is selected as discussed below. Once this selection has been made, the shear τ_w (or the ratio of τ_w to the normal stress $\rho_p \nu g h \cos\theta$) may be computed from measured quantities $h(x)$, $\nu(x)$ and the total mass flow rate. The ratio $\tau_w/\rho_p \nu g h \cos\theta$ is called the friction coefficient, μ.

The accuracy of the determination of τ_w from equation (1) depends on the accuracy with which the variables can be measured. Perhaps the largest source of uncertainty comes from the measurement of h, particularly for flows of very low densities. This measurement in turn influences the values of the density ν. Under these conditions we have allowed for errors as high as ±30 percent. Fortunately, the product νh, which enters the computation of τ_w, is obtained directly from the trapped material and this quantity is, therefore, more accurately assessed than either ν or h alone. The terms involving the derivatives dh/dx and $d\nu/dx$ were, in all cases, small compared to the body force term, and did not materially contribute to errors in the computation of τ_w.

A comment should be added regarding the constant β. A value of $\beta = 2$ results from the assumption that the shear against the side walls is equal to that at the channel bottom. A value of $\beta = 0$, on the other hand, represents frictionless side walls. For the granular materials tested, the data suggest a value for β somewhere in between. Furthermore, at very low velocities, one would expect the shear to correspond to that produced by Coulomb friction between two solids. With this consideration in mind, β was selected to be unity. Fortunately the selection of β is not critical to the determination of the shear τ_w, since the chute was designed to be sufficiently wide with the express purpose of minimizing the wall friction effect.

V Experimental Results

A large number of experiments were conducted using two sizes of glass beads measuring 0.3 mm and 3.0 mm in diameter. The material properties of these glass beads are given in Table 1. For each material the channel was set at a number of different angles and for each angle a range of mass flow rates and flow depths were covered. The flow was controlled by gates at the inlet section and in some cases also at the discharge section of the channel.

The experimental results may be presented in terms of a number of possible dimensionless parameters. The two parameters which seem to be particularly suitable are the friction coefficient $(\tau_w/\rho_p \nu g h \cos\theta)$ and the Froude number $U/(gh\cos\theta)^{1/2}$. Additional parameters, such as h/d, may be considered when results for different sizes of particles are compared.

Before considering the variations in the friction coefficient with various parameters, it is useful to examine the effects of the density on the flow characteristics. To obtain further insight into the relation between the density and the flow characteristics, one may define a quantity

$$N = \frac{\nu h}{d}$$

One may think of N as the number of particle layers in the flow. By plotting N against ν, one can examine the extent to which the dilation of the material depends on the number of layers of particles in the flow. Fig. 3 is such a graph and contains the data for both the 0.3 and 3 mm glass beads. Note that for both sizes, the depth-averaged solids fraction, ν, is essentially constant for flows with N greater than approximately four. On the other hand, when N decreases below 4 the material dilates substantially. These data appear to be almost independent of shear rate.

The relationship between the friction coefficient, μ, and the Froude number is shown in Fig. 4. The data for both sizes of glass beads indicate that the friction coefficient remains fairly constant up to a certain Froude number. Beyond that, the data shows a rapid increase in μ with a further increase in the Froude number. Even though the individual points may be subject to large errors, as pointed out earlier, the measured increase in friction coefficient is so large as to leave no doubt about the reality of this sudden rise. The deviation from this constant value occurs at different Froude numbers for the two sizes. The smaller glass beads show a more gradual increase with Froude number than the larger glass beads, but it is apparent that a definite deviation from the Coulomb friction coefficient occurs. For both sizes of beads the value of the friction coefficient at low Froude numbers is essentially equal to the Coulomb friction coefficient for glass beads against aluminum, which implies Coulomb friction governs these flows in this range. At higher flow velocities, the increase in the friction coefficient represents a deviation from this simple Coulomb behavior. This behavior of the friction coefficient has an interesting consequence. It implies that a solid material in granular form may reach uniform flow (non-accelerating flow) at various chute angles.

The question arises as to the mechanism responsible for the increase in the friction coefficient at high Froude numbers. At these high Froude numbers it is observed that the particles are highly agitated and contact with the wall consists of collisions

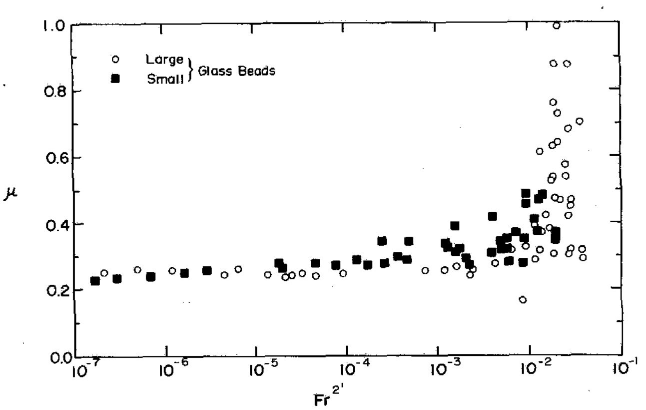

Fig. 5 The variation of the friction coefficient, μ, as a function of Froude Number, $Fr^{2\,\prime} = Fr^2(d/h)^2$. (Both sizes of glass beads are shown.)

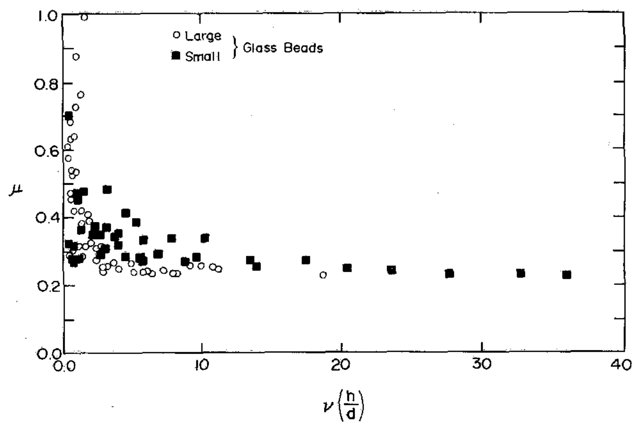

Fig. 6 The variation of the friction coefficient, μ, as a function of $\nu(h/d)$. (Both sizes of glass beads are shown.)

rather than sliding. These different means for the exchange of momentum might explain the deviation from Coulomb friction seen in these flows. The precise mechanism responsible for an increase in the effective friction angle, however, is unclear. The interaction between the granules and the channel surface is likely to be quite complex and may depend on additional parameters including the size of the particles, the thickness of the channel bottom, the materials involved and the condition of the surfaces.

As we have already observed, the experimental data of the friction coefficient for the two different sizes of beads fall on two different curves when plotted as a function of the Froude number. This fact seems to indicate that a geometrical factor such as h/d is influencing the results. In an attempt to include this factor in a simple way, a modified Froude number is introduced.

$$Fr' = Fr(d/h)$$

The results for both sizes of glass beads are shown in Fig. 5 in which the friction coefficient is plotted against this new Froude number. The two sets of data appear to fall on approximately the same curve. The generalization of this result will depend on future experiments with a greater variety of sizes and materials. Nevertheless, the data in Fig. 5 indicate the possibility of obtaining reasonable correlations by means of simple modified parameters.

So far the friction coefficient, μ, has been presented as a function of the Froude numbers. One might also consider the average density ν as a parameter. In order to account also for the factor h/d, several combinations of ν and h/d were tried and a rather acceptable correlation was established between ν and $\nu(h/d)$, as shown in Fig. 6. In general, low densities coin-

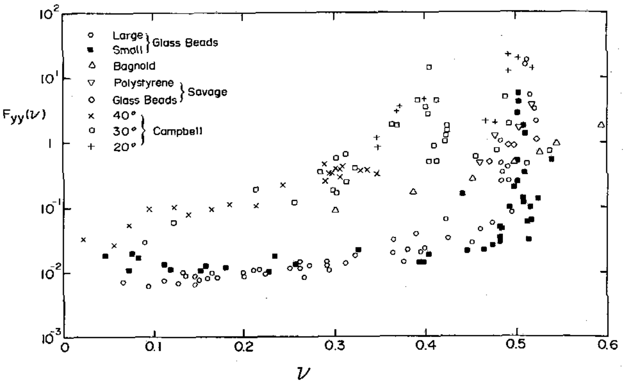

Fig. 7 The variation of $F_{xy}(\nu)$ as a function of ν as compared with previous works

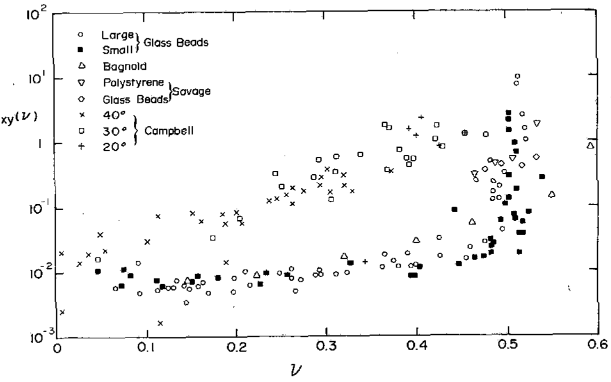

Fig. 8 The variation of $F_{yy}(\nu)$ as a function of ν, as compared with previous works

cide with high Froude numbers. Therefore, the friction coefficient reaches its largest values as $\nu(h/d)$ becomes small.

VI Implications for the Constitutive Relations

In a model originally proposed by Bagnold (1954, 1956, 1966) and further developed by Lun et al. (1984), Jenkins and Savage (1983), as well as others, the stresses in a rapidly flowing granular material may be expressed in the form so that

$$\sigma_{ij} = \rho_p F_{ij}(\nu) d^2 (du/dy)^2 \tag{2}$$

where the functions F_{ij} depend only on the solid fraction ν. The available experimental data do not allow accurate computations or local determinations of these functions. However, since the formulation of stresses in terms of the functions F_{ij} represents a very fundamental aspect in the understanding of granular flows, an attempt to obtain some rough estimates of those functions seems justified. Therefore, since details of the velocity profile and any possible slip at the wall have not been measured, the velocity gradient du/dy will be characterized by U/h where U is the average velocity and h the depth of flow. The stress σ_{yy} and σ_{xy} are taken to be the pressure and shear on the channel bottom and the corresponding density functions are given the symbols F_{yy} and F_{xy}.

The function F_{xy} computed in this way for the present study is shown in Fig. 7. The data for the two sizes of glass beads fall essentially on the same curve. The data from the experiments by Bagnold (1954, 1956, 1966) and by Savage and Sayed (1982) are shown which were taken using Couette flow devices. Considering the rough approximations used in the computation of F_{xy}, the data from these experiments correlate rather well with the present results and adds further support to the significance of Bagnold's original formulation. The data for the normal stress function F_{yy} are shown in Fig. 8 together with those of Bagnold and Savage. The same conclusions may be drawn about the normal stress function.

It should be mentioned that values for the functions F_{xy} and F_{yy} may also be derived from the computational work of Campbell (1982) and others. The values derived by Campbell, although showing the same trends, are numerically higher than those derived experimentally. These computational works, however, are based on a number of assumptions concerning the details of the collision mechanics which, at this time, cannot be compared to the actual physical conditions. The data developed by Campbell (1982) for the density functions, F_{xy} and F_{yy}, are also shown in Figs. 7 and 8.

VIII Summary and Conclusions

An extensive experimental study of the flow of granular materials in an open channel was performed. The materials involved in the study were glass beads of two different sizes. A technique was developed which allowed the measurement of the average density of the flowing material. The results clearly show that for the materials studied in this work the friction coefficient, μ, is not a constant as in the case of two solids in sliding contact but rather that the friction coefficient increases with increases in the Froude number. This result implies that for the flows of granular materials in open channels, it may be possible to obtain uniform flow for a range of different angles of channel inclination.

The experimental data was also used to assess the validity of the analytic expressions proposed by Bagnold for the stresses developed in a granular material flow. In particular, Bagnold's expressions for the shear and normal force, which define the functions F_{xy} and F_{yy} as functions of only the density, were tested. The results support such a relationship and are in qualitative agreement with other experimental and theoretical works for simple shear flows of granular materials.

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