## 3.5 Neutron transport theory

The first simplification of *neutron transport theory* is to assume that the range of neutron energies can be discretized into a small number of energy ranges (sometimes, as has been described in the preceding section, the even more radical assumption is made that all neutrons have the same energy). Then the heart of neutron transport theory is a neutron continuity equation known as the *neutron transport equation* that simply represents the neutron gains and losses for an arbitrary control volume, $V$, within the reactor for each of the ranges of neutron energies being considered. In evaluating this neutron balance for each of the energy ranges it is necessary to account for:

[A] The rate of increase of those neutrons within the volume $V$.

[B] The rate of appearance of those neutrons in $V$ as a result of flux through the surface of the volume $V$.

[C] The loss of those neutrons as a result of absorption (and as a result of scattering to an energy level outside of the entire range of discretized energies).

[D] The rate of appearance of those neutrons that, as a result of a scattering interaction, now have energies of the magnitude being evaluated.

[E] The rate of production of those neutrons in $V$, most importantly by fission.

These alphabetical labels will be retained when each of these individual terms is considered in the analysis that follows.

The second simplification, mentioned earlier, recognizes that the angular variations in the neutron flux are rarely of first order importance. Hence non-isotropic details can be laid aside and the neutron flux can be integrated over the angular orientation, $\Omega_j$, as described in equations 4 and 5 of section 3.2. When this integration is performed on the neutron transport equation in order to extract an equation for the integrated neutron flux, $\phi(x_i, t, E)$, the result takes the following form (Glasstone and Sesonske 1981, Duderstadt and Hamilton 1976):

$$\frac{1}{\bar{u}}\frac{\partial \phi}{\partial t} + \frac{\partial J_j}{\partial x_j} + \Sigma_a \phi = \int_0^\infty \Sigma_s(E' \rightarrow E)\phi(x_i, t, E')dE' + S(x_i, t, E) \qquad (1)$$

This is known as the *neutron continuity equation.* The five terms each represent a contribution to the population (per unit volume) of neutrons of energy $E$ at the location $x_i$ and the time $t$; specifically:

[A] The first term is the rate of increase of neutrons in that unit volume.

[B] The second term is the flux of neutrons out of that unit volume.

[C] The third term is the rate of loss of neutrons due to absorption.

[D] The fourth term is the rate of increase of neutrons of energy $E$ due to scattering where the energy before the scattering interaction was $E'$. Consequently an integration over all possible previous energies, $E'$, must be performed.

[E] The fifth term is the rate of production of neutrons of energy $E$ within the unit volume due to fission, $S(x_i, t, E)$.

Consequently the following nomenclature pertains in equation 1: $\phi(x_i, t, E)$ and $J_j(x_i, t, E)$ are the angle-integrated flux and current density as defined by equations 4 and 5 of section 3.2, $\bar{u}$ represents the magnitude of the neutron velocity (assumed isotropic), $\Sigma_a(x_i, E)$ is the macroscopic cross-section at location $x_i$ for collisions in which neutrons of energy, $E$, are absorbed, $\Sigma_s(E' \rightarrow E)$ is the macroscopic cross-section for scattering of neutrons of energy $E'$ to energy $E$, and $S(x_i, t, E)$ is the rate of production in a unit volume at $x_i$ and $t$ of neutrons of energy $E$.

Assuming that the macroscopic cross-sections and the source term are given, equation 1 is the equation that determines the population of neutrons for each energy level $E$ as a function of position $x_i$ and time $t$. Ideally this equation should be solved for the neutron flux, $\phi(x_i, t, E)$. However, there remains a problem in that the equation involves two unknown functions, $\phi(x_i, t, E)$ and $J_j(x_i, t, E)$, a problem that was further complicated by the integration over the angle. Specifically, whereas $\varphi(x_i, t, E, \Omega_j)$ and $J_j^*(x_i, t, E, \Omega_j)$ are simply related by equation 3 of section 3.2, the functions, $\phi(x_i, t, E)$ and $J_j(x_i, t, E)$, defined respectively by equations 4 and 5 of section 3.2, are not so easily related.

To proceed with a solution, another relation between $\phi(x_i, t, E)$ and $J_j(x_i, t, E)$ must be found. One simple way forward is to heuristically argue that in many transport processes (for example the conduction of heat), the concentration (in this case $\phi$) and the flux (in this case $J_j$) are simply connected by a relation known as Fick's law in which the flux is proportional to the gradient of the concentration, the factor of proportionality being a diffusion coefficient. This assumption or approximation is made here by heuristically declaring that

$$J_j(x_i, t, E) = -D(x_i)\frac{\partial \phi(x_i, t, E)}{\partial x_j} \tag{2}$$

where $D$ is a diffusion coefficient that may be a function of position. This diffusive process could be viewed as the effective consequence of neutrons undergoing multiple scattering interactions just as heat diffusion is the effective consequence of molecules undergoing multiple interactions. One of the modern computational approaches to neutron transport known as the Monte Carlo method (see section 3.11) utilizes this general consequence.

Fick's law will be the model that will be the focus here. However, it is valuable to point out that Fick's law for neutrons can also be derived from the basic conservation laws in the following way. Returning to the neutron continuity principle, one can propose an expansion for the neutron flux, $\varphi$, that includes the angle-integrated average used above plus a perturbation term that is linear in the angle $\Omega_j$. Assuming that this second term is small (that the flux is only weakly dependent on the angle), one can then establish the equation for this linear perturbation term that emerges from the neutron continuity principle. Making some further assumptions (neglect of the time dependent term, assumption of isotropic source term), the result that emerges from this perturbation analysis is:

$$\frac{1}{3}\frac{\partial \phi}{\partial x_j} + \Sigma_{tr} J_j = 0 \tag{3}$$

where $\Sigma_{tr}$ is called the *macroscopic transport cross-section* and is given by $\Sigma_{tr} = \Sigma_a + \Sigma_s - \mu\Sigma_s$ where $\mu$ is the cosine of the average scattering angle. (For further detail and a rigorous derivation of these relations the reader should consult texts such as Glasstone and Sesonske 1981 or Duderstadt and Hamilton 1976). Comparing equation 3 with equation 2 it can be observed that $J_j$ and $\phi$ do, indeed, connect via Fick's law and that the *neutron diffusion coefficient*, $D(x_i)$, is given by

$$D(x_i) = \frac{1}{3\Sigma_{tr}} \tag{4}$$

Equation 3 can then be used to substitute for $J_j$ in equation 1 and thus generate an equation for the single unknown function, $\phi(x_i, t, E)$.

Computational methods based on the assumption of equation 2 are known as diffusion theories and these will be the focus of the sections that follow.