Example of Rapid Flow Equations

Later, the work of Savage and Jeffrey (1981) and Jenkins and Savage (1983) saw the beginning of a more rigorous application of kinetic theory methods to rapid granular flows and there is now an extensive literature on the subject (see, for example, Gidaspow 1994). The kinetic theories may be best exemplified by quoting the results of Lun *et al.* (1984) who attempted to evaluate both the collisional and streaming contributions to the stress tensor (since momentum is transported both by the collisions of finite-sized particles and by the motions of the particles). In addition to the continuity and momentum equations, equations (Npg1) and (Npg2), an *energy equation* must be constructed to represent the creation, transport and dissipation of granular heat; the form adopted is

$$\frac{3}{2}\rho_S\alpha\frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\sigma_{ji} - \Gamma \tag{Npk1}$$

where T is the granular temperature, q_i is the granular heat flux vector, and Γ is the rate of dissipation of granular heat into thermodynamic heat per unit volume. Note that this represents a balance between the granular heat stored in a unit volume (the lefthand side), the conduction of granular heat into the unit volume (first term on RHS), the generation of granular heat (second term on RHS) and the dissipation of granular heat (third term on RHS).

Most of the kinetic theories begin in this way but vary in the expressions obtained for the stress/strain relations, the granular heat flux and the dissipation term. As an example we quote here the results from the kinetic theory of Lun *et al.* (1984) that have been subsequently used by a number of authors. Lun *et al.* obtain a stress tensor related to the granular temperature, T (equation (Npj1)), by

$$\sigma_{ij} = \left(\rho_S g_1 T - \frac{4\pi^{\frac{1}{2}}}{3}\rho_S\alpha^2(1+\epsilon)g_0 T^{\frac{1}{2}}\frac{\partial u_i}{\partial x_i}\right)\delta_{ij}$$
$$-2\rho_S D g_2 T^{\frac{1}{2}}\left(\frac{1}{2}(u_{ij}+u_{ji}) - \frac{1}{3}u_{kk}\delta_{ij}\right) \tag{Npk2}$$

an expression for the granular heat flux vector,

$$q_i = -\rho_S D\left(g_3 T^{\frac{1}{2}}\frac{\partial T}{\partial x_i} + g_4 T^{\frac{3}{2}}\frac{\partial \alpha}{\partial x_i}\right) \tag{Npk3}$$

and an expression for the rate of dissipation of granular heat,

$$\Gamma = \rho_S g_5 T^{\frac{3}{2}}/D \tag{Npk4}$$

where $g_0(\alpha)$, the radial distribution function, is chosen to be

$$g_0 = (1-\alpha/\alpha^*)^{-2.5\alpha^*} \tag{Npk5}$$

and α^* is the maximum shearable solids fraction. In the expressions (Npk2), (Npk3), and (Npk4), the quantities g_1, g_2, g_3, g_4, and g_5, are functions of α and ϵ as follows:

$$g_1(\alpha,\epsilon) = \alpha + 2(1+\epsilon)\alpha^2 g_0$$

$$g_2(\alpha,\epsilon) = \frac{5\pi^{\frac{1}{2}}}{96}\left(\frac{1}{\eta(2-\eta)g_0} + \frac{8(3\eta-1)\alpha}{5(2-\alpha)} + \frac{64\eta\alpha^2 g_0}{25}\left(\frac{(3\eta-2)}{(2-\eta)} + \frac{12}{\pi}\right)\right)$$

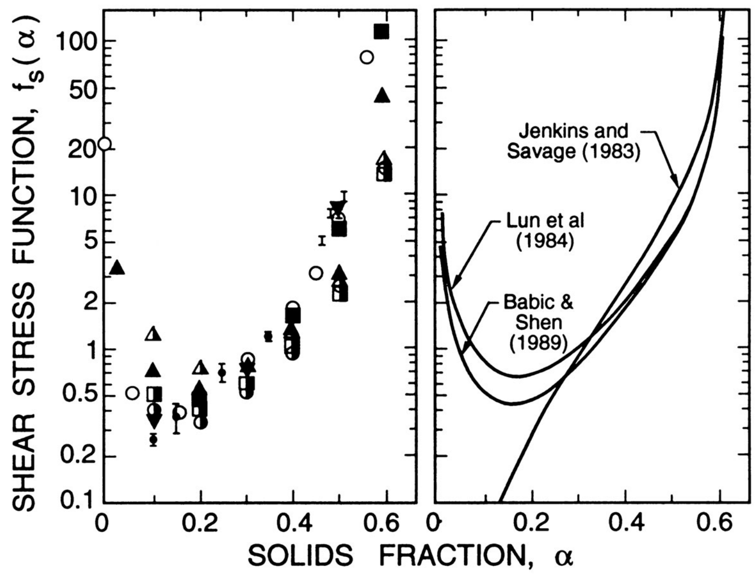

Figure 1: Left: the shear stress function, $f_s(\alpha)$, from the experiments of Savage and Sayed (1984) with glass beads (symbol I) and various computer simulations (open symbols: with hard particle model; solid symbols: with soft particle model; half solid symbols: with Monte Carlo methods). Right: Several analytical results. Adapted from Campbell (1990).

$$g_3(\alpha,\epsilon) = \frac{25\pi^{\frac{1}{2}}}{16\eta(41-33\eta)}\left(\frac{1}{g_0} + 2.4\eta\alpha(1-3\eta+4\eta^2)\right.$$
$$\left. + \frac{16\eta^2\alpha^2 g_0}{25}\left(9\eta(4\eta-3)+4(41-33\eta)/\pi\right)\right)$$
$$g_4(\alpha,\epsilon) = \frac{15\pi^{\frac{1}{2}}(2\eta-1)(\eta-1)}{4(41-33\eta)}\left(\frac{1}{\alpha g_0}+2.4\eta\right)\frac{d}{d\alpha}(\alpha^2 g_0)$$
$$g_5(\alpha,\epsilon) = \frac{48\eta(1-\eta)\alpha^2 g_0}{\pi^{\frac{1}{2}}} \qquad \text{(Npk6)}$$

where $\eta = (1+\epsilon)/2$.

For two-dimensional shear flows in the (x, y) plane with a shear $\partial u/\partial y$ and no acceleration in the x direction the Lun *et al.* relations yield stresses given by:

$$\sigma_{xx} = \sigma_{yy} = \rho_S g_1 T \quad ; \quad \sigma_{xy} = -\rho_S D g_2 T^{\frac{1}{2}}\frac{\partial u}{\partial y} \qquad \text{(Npk7)}$$

in accord with the expressions (Npj2). They also yield a granular heat flux component in the y direction given by:

$$q_y = \rho_S D\left(g_3 T^{\frac{1}{2}}\frac{\partial T}{\partial y} + g_4 T^{\frac{3}{2}}\frac{\partial\alpha}{\partial y}\right) \qquad \text{(Npk8)}$$

These relations demonstrate the different roles played by the quantities g_1, g_2, g_3, g_4, and g_5: g_1 determines the normal kinetic pressure, g_2 governs the shear stress or *viscosity*, g_3 and g_4 govern the diffusivities controlling the conduction of granular heat from regions of differing temperature and density and g_5 determines the granular dissipation. While other kinetic theories may produce different specific expressions for these quantities, all of them seem necessary to model the dynamics of a rapid granular flow.

Figure 1 shows typical results for the shear stress function, $f_s(\alpha)$. The lefthand graph includes the data of Savage and Sayed (1984) from shear cell experiments with glass beads as well as a host of computer simulation results using both hard and soft particle models and both mechanistic and Monte Carlo methods. The righthand graph presents some corresponding analytical results. The stress states to the left of the minima in these figures are difficult to observe experimentally, probably because they are unstable in most experimental facilities.

In summary, the governing equations, exemplified by equations (Npg1), (Npg2) and (Npk1) must be solved for the unknowns, α, T and the three velocity components, u_i given the expressions for σ_{ij}, q_i and Γ and the physical constants D, ρ_S, ϵ, α^* and gravity g_k.

It was recognized early during research into rapid granular flows that some modification to the purely collisional kinetic theory would be needed to extend the results towards lower shear rates at which frictional stresses become significant. A number of authors explored the consequences of heuristically adding frictional terms to the collisional stress tensor (Savage 1983, Johnson *et al.*, 1987, 1990) though it is physically troubling to add contributions from two different flow regimes.