Bubble Natural Frequencies

In this and the sections that follow we will consider the response of a bubble to oscillations in the prevailing pressure. We begin with an analysis of bubble natural frequencies in the absence of thermal effects and liquid compressibility effects. Consider the linearized dynamic solution of equation (Ngd2) when the pressure at infinity consists of a mean value, $\bar{p}_\infty$, upon which is superimposed a *small* oscillatory pressure of amplitude, $\tilde{p}$, and radian frequency, ω, so that

$$p_\infty = \bar{p}_\infty + Re\{\tilde{p}e^{j\omega t}\} \tag{Ngj1}$$

The linear dynamic response of the bubble will be represented by

$$R = R_e[1 + Re\{\varphi e^{j\omega t}\}] \tag{Ngj2}$$

where R_e is the equilibrium size at the pressure, $\bar{p}_\infty$, and the bubble radius response, φ, will in general be a complex number such that $R_e|\varphi|$ is the amplitude of the bubble radius oscillations. The phase of φ represents the phase difference between p_∞ and R.

For the present we shall assume that the mass of gas in the bubble, m_G, remains constant. Then substituting equations (Ngj1) and (Ngj2) into equation (Ngd2), neglecting all terms of order $|\varphi|^2$ and using the equilibrium condition (Nge1) one finds

$$\omega^2 - j\omega\frac{4\nu_L}{R_e^2} + \frac{1}{\rho_L R_e^2}\left\{\frac{2S}{R_e} - 3kp_{Ge}\right\} = \frac{\tilde{p}}{\rho_L R_e^2 \varphi} \tag{Ngj3}$$

where, as before,

$$p_{Ge} = \bar{p}_\infty - p_V + \frac{2S}{R_e} = \frac{3m_G T_B \mathcal{R}_G}{4\pi R_e^3} \tag{Ngj4}$$

It follows that for a given amplitude, $\tilde{p}$, the maximum or peak response amplitude occurs at a frequency, ω_p, given by the minimum value of the spectral radius of the left-hand side of equation (Ngj3):

$$\omega_p = \left\{\frac{(3kp_{Ge} - 2S/R_e)}{\rho_L R_e^2} - \frac{8\nu_L^2}{R_e^4}\right\}^{\frac{1}{2}} \tag{Ngj5}$$

or in terms of $(\bar{p}_\infty - p_V)$ rather than p_{Ge}:

$$\omega_p = \left\{\frac{3k(\bar{p}_\infty - p_V)}{\rho_L R_e^2} + \frac{2(3k-1)S}{\rho_L R_e^3} - \frac{8\nu_L^2}{R_e^4}\right\}^{\frac{1}{2}} \tag{Ngj6}$$

At this peak frequency the amplitude of the response is, of course, inversely proportional to the damping:

$$|\varphi|_{\omega=\omega_p} = \frac{\tilde{p}}{4\mu_L\left\{\omega_p^2 + \frac{4\nu_L^2}{R_e^4}\right\}^{\frac{1}{2}}} \tag{Ngj7}$$

It is also convenient for future purposes to define the natural frequency, ω_n, of oscillation of the bubbles as the value of ω_p for zero damping:

$$\omega_n = \left\{\frac{1}{\rho_L R_e^2}\left\{3k(\bar{p}_\infty - p_V) + 2(3k-1)\frac{S}{R_e}\right\}\right\}^{\frac{1}{2}} \tag{Ngj8}$$

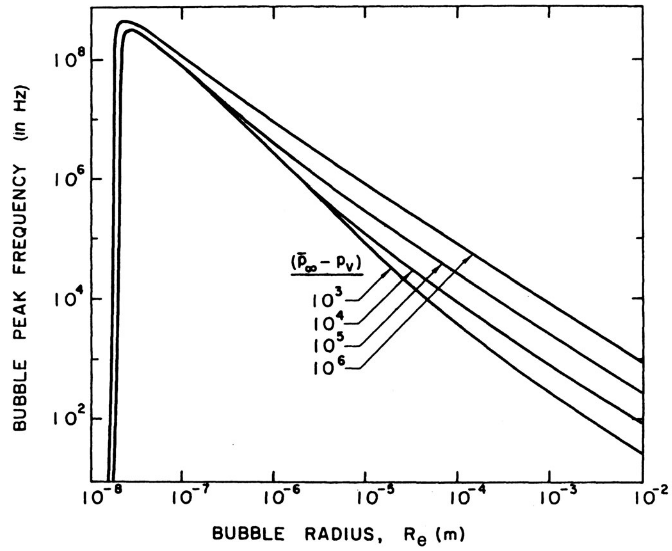

Figure 1: Bubble resonant frequency in water at $300°K$ ($S = 0.0717$, $\mu_L = 0.000863$, $\rho_L = 996.3$) as a function of the radius of the bubble for various values of $(\bar{p}_\infty - p_V)$ as indicated.

The connection with the stability criterion of section (Nge) is clear when one observes that no natural frequency exists for tensions $(p_V - \bar{p}_\infty) > 4S/3R_e$ (for isothermal gas behavior, $k = 1$); stable oscillations can only occur about a stable equilibrium.

Note from equation (Ngj6) that ω_p is a function only of $(\bar{p}_\infty - p_V)$, R_e, and the liquid properties. A typical graph for ω_p as a function of R_e for several $(\bar{p}_\infty - p_V)$ values is shown in figure 1 for water at $300°K$ (S $= 0.0717$, $\mu_L = 0.000863$, $\rho_L = 996.3$). As is evident from equation (Ngj6), the second and third terms on the right-hand side dominate at very small R_e and the frequency is almost independent of $(\bar{p}_\infty - p_V)$. Indeed, no peak frequency exists below a size equal to about $2\nu_L^2 \rho_L / S$. For larger bubbles the viscous term becomes negligible and ω_p depends on $(\bar{p}_\infty - p_V)$. If the latter is positive, the natural frequency approaches zero like R_e^{-1}. In the case of tension, $p_V > \bar{p}_\infty$, the peak frequency does not exist above $R_e = R_c$.

For typical nuclei found in water (1 to 100 μm) the natural frequencies are of the order, 5 to $25kHz$. This has several important practical consequences. First, if one wishes to cause cavitation in water by means of an imposed acoustic pressure field, then the frequencies that will be most effective in producing a substantial concentration of large cavitation bubbles will be in this frequency range. This is also the frequency range employed in magnetostrictive devices used to oscillate solid material samples in water (or other liquid) in order to test the susceptibility of that material to cavitation damage (Knapp *et al.* 1970). Of course, the oscillation of the nuclei produced in this way will be highly nonlinear and therefore peak response frequencies will be significantly lower than those given above.

There are two important footnotes to this linear dynamic analysis of an oscillating bubble. First, the assumption that the gas in the bubble behaves polytropically is a dubious one. Prosperettti (1977) has analysed the problem in detail with particular attention to heat transfer in the gas and has evaluated the effective polytropic exponent as a function of frequency. Not surprisingly the polytropic exponent increases from unity at very low frequencies to γ at intermediate frequencies. However, more unexpected behaviors develop at high frequencies. At the low and intermediate frequencies, the theory is largely in agreement with Crum's (1983) experimental measurements. Prosperetti, Crum, and Commander (1988) provide a useful summary of the issue.

A second, related concern is the damping of bubble oscillations. Chapman and Plesset (1971) presented

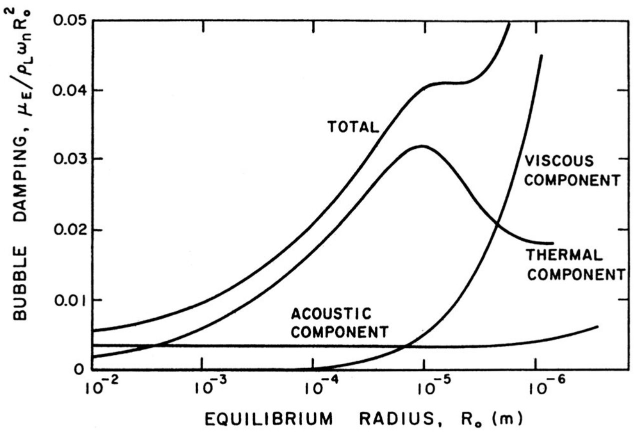

Figure 2: Bubble damping components and the total damping as a function of the equilibrium bubble radius, R_e, for water. Damping is plotted as an *effective* viscosity, μ_e, nondimensionalized as shown (from Chapman and Plesset 1971).

a summary of the three primary contributions to the damping of bubble oscillations, namely that due to liquid viscosity, that due to liquid compressibility through acoustic radiation, and that due to thermal conductivity. It is particularly convenient to represent the three components of damping as three additive contributions to an effective liquid viscosity, μ_e, that can then be employed in the Rayleigh-Plesset equation in place of the actual liquid viscosity, μ_L :

$$\mu_e = \mu_L + \mu_t + \mu_a \tag{Ngj9}$$

where the *acoustic* viscosity, μ_a, is given by

$$\mu_a = \frac{\rho_L \omega^2 R_e^3}{4c_L} \tag{Ngj10}$$

where c_L is the velocity of sound in the liquid. The *thermal* viscosity, μ_t, follows from the analysis by Prosperettti (1977) mentioned in the last paragraph (see also Brennen 1995). The relative magnitudes of the three components of damping (or *effective* viscosity) can be quite different for different bubble sizes or radii, R_e. This is illustrated by the data for air bubbles in water at 20°C and atmospheric pressure that is taken from Chapman and Plesset (1971) and reproduced as figure 2.