Solution to Problem 241D

The actual cross-section of the duct as a function of x is

$$b\left[h_0 + Hx^k\right]$$

but due to the displacement thickness of the boundary layers on the walls, $\delta_D(x)$, the effective cross-sectional area of the duct is

$$bh_0 + bHx^k - 2b\delta_D$$

Using the Falkner-Skan solutions for $U = Cx^m$ we find that the displacement thickness

$$\delta_D = \left\{\frac{4\nu x^{(1-m)}}{C}\right\}^{\frac{1}{2}} \int_0^\infty \left\{1 - \frac{dF}{d\eta}\right\} d\eta$$

For convenience we denote the number that the integral represents by

$$I(m) = \int_0^\infty \left\{1 - \frac{dF}{d\eta}\right\} d\eta$$

where, since $F(\eta)$ depends on m, the value of $I(m)$ changes with m.

If the longitudinal pressure gradient is to be zero, we must have U constant and therefore $m = 0$ so that the appropriate boundary layer solution is the Blasius laminar boundary layer solution for which

$$\delta_D = 1.72\left\{\frac{\nu x}{U}\right\}^{\frac{1}{2}}$$

But conservation of mass requires that U multiplied by the effective cross-sectional area must be a constant and since U is a constant in this case, it follows that the effective cross-sectional area must be a constant. The only way that this can be true is if

$$Hx^k = 2\delta_D$$

and it therefore follows that, in this case,

$$k = 1/2 \qquad \text{and} \qquad H = 3.44(\nu/U)^{\frac{1}{2}}$$