Solution to Problem 150F

Part [1]

With the prescription of the flow in this problem, the Navier-Stokes equations become

$$-\rho\frac{u_\theta^2}{r} = -\frac{dp}{dr}$$

$$0 = \mu\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2}\right)$$

The note at the end of the problem provides the solution to the differential equation,

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \frac{X}{r^2} = 0$$

namely

$$X = Ar + \frac{B}{r}$$

where A and B are integration constants. In the present problem this yields

$$u_\theta = Ar + \frac{B}{r}$$

We now apply the boundary conditions to determine the values of A and B. At $r = a$ (the surface of the inner, stationary cylinder) $u_\theta = 0$ by the no-slip condition, so that

$$0 = Aa + \frac{B}{a} \quad\Longrightarrow\quad B = -Aa^2$$

Also at $r = b$ (the surface of the outer, rotating cylinder) $u_\theta = \Omega b$, where Ω is the angular velocity of the outer cylinder, so that

$$\Omega b = Ab + \frac{B}{b} = A\left(b - \frac{a^2}{b}\right) \quad\Longrightarrow\quad A = \frac{\Omega b}{b^2 - a^2}$$

Substituting these expressions for A and B into the flow solution yields

$$u_\theta = \frac{\Omega b^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)$$

Part [2]

Using this solution the first equation yields

$$\frac{dp}{dr} = \rho\frac{u_\theta^2}{r} = \rho\frac{\Omega^2 b^4}{(b^2 - a^2)^2}\left(r - 2\frac{a^2}{r} + \frac{a^4}{r^3}\right)$$

and integrating this yields

$$p(r) = \rho\frac{\Omega^2 b^4}{(b^2 - a^2)^2}\left(\frac{1}{2}r^2 - 2a^2\ln r - \frac{a^4}{2r^2}\right) + C$$

where C is an integration constant. This can be used to find the pressure difference between the surfaces of the two cylinders, namely

$$p(b) - p(a) = \left[\rho\frac{\Omega^2 b^4}{(b^2 - a^2)^2}\left(\frac{1}{2}b^2 - 2a^2\ln b - \frac{a^4}{2b^2}\right)\right] - \left[\rho\frac{\Omega^2 b^4}{(b^2 - a^2)^2}\left(\frac{1}{2}a^2 - 2a^2\ln a - \frac{a^2}{2}\right)\right]$$

which simplifies to

$$p(b) - p(a) = \rho\frac{\Omega^2 b^4}{(b^2 - a^2)^2}\left[\frac{1}{2}\left(b^2 - a^2\right) - 2a^2\left(\ln b - \ln a\right) - \frac{a^2}{2}\left(\frac{a^2}{b^2} - 1\right)\right]$$

Part [3]

The definition of the shear stress on the wall is

$$\sigma|_{\text{wall}} = \mu \left.\frac{du_\theta}{dr}\right|_{\text{wall}}$$

Calculating du_θ/dr from the solution to the flow

$$\frac{du_\theta}{dr} = \frac{\Omega b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

For the inner cylinder

$$\sigma_{r=a} = 2\mu\frac{\Omega b^2}{b^2 - a^2}$$

and for the outer cylinder

$$\sigma|_{r=b} = \mu\frac{\Omega b^2}{b^2 - a^2}\left(1 + \frac{a^2}{b^2}\right)$$

Part [4]

The power required to rotate the outer cylinder is given by

$$P = Fu$$

where F is the force necessary to rotate the cylinder and u is the speed with which the cylinder is rotating. In this problem

$$P = \sigma|_{r=b}\, A_{\text{cylinder}}\; u_\theta|_{r=b}$$

If the length of the outer cylinder is L, then

$$A_{\text{cylinder}} = 2\pi b L$$

Evaluating the stress and velocity at $r = b$ and substituting yields

$$P = \left[\mu\frac{\Omega b^2}{b^2 - a^2}\left(1 + \frac{a^2}{b^2}\right)\right](2\pi b L)(\Omega b)$$

which simplfies to

$$P = 2\pi L\mu\frac{\Omega^2 b^4}{b^2 - a^2}\left(1 + \frac{a^2}{b^2}\right)$$