Solution to Problem 323A:

The velocity, pressure, density, temperature and Mach number upstream of the injection point are denoted by u, p, ρ, T, and M respectively. The velocity, pressure, density, temperature and Mach number downstream of the injection point are denoted by $u + du$, $p + dp$, $\rho + d\rho$, $T + dT$, and $M + dM$ respectively. Then applying the basic equations before and after the injection point where the mass rate of injection is $\rho u di$:

[A] Continuity yields

$$(\rho + d\rho)(u + du) \;=\; \rho u + \rho u di \tag{1}$$

[B] Since the injected stream has no momentum in the direction of the main stream, the momentum equation yields

$$p - (p + dp) \;=\; (\rho + d\rho)(u + du)^2 - \rho u^2 \tag{2}$$

and using $p = \rho \mathcal{R} T$ and $u^2 = \gamma \mathcal{R} T\ M^2$ this gives

$$-\frac{1}{\gamma M^2}\frac{dp}{p} \;=\; \frac{du}{u} + di \tag{3}$$

[C] Since both incoming streams have the same total temperature or total enthalpy, the downstream stream must have the same total enthalpy. Therefore the energy equation yields

$$c_p dT + u\ du \;=\; 0 \quad \text{or} \quad \frac{1}{(\gamma - 1)M^2}\frac{dT}{T} + \frac{du}{u} \;=\; 0 \tag{4}$$

using $c_p = \gamma \mathcal{R}/(\gamma - 1)$ and $u^2 = \gamma \mathcal{R} T M^2$.

[D] The perfect gas law, $p = \rho \mathcal{R} T$, yields

$$\frac{dp}{p} \;=\; \frac{d\rho}{\rho} + \frac{dT}{T} \tag{5}$$

Then by elimination

$$\frac{dT}{T} \;=\; (1 - \gamma M^2)\frac{du}{u} - (1 + \gamma M^2)di \;=\; -(\gamma - 1)M^2\frac{du}{u} \tag{6}$$

and

$$\frac{du}{u} \;=\; \frac{(1 + \gamma M^2)}{(1 - M^2)}di \tag{7}$$