

Solution to Problem 312C:

Heat is being added to the steady, frictionless flow of a perfect gas (ratio of specific heats, γ) in a pipe of constant, uniform cross-sectional area. The speed of sound and Mach number of the flow are denoted by c and M respectively and vary with position, x , measured along the pipe. If the rate of heat addition is Q per unit time per unit length of the pipe and the mass flow rate of gas is denoted by m we seek to find an expression for dM/dx in terms of Q , m , γ , c and M .

The heat added to unit mass in a distance, dx , is Qdx/m and this must be equal to the increase in total enthalpy for a unit mass so

$$\frac{Qdx}{m} = c_p dT + u du \quad (1)$$

and therefore from the energy equation

$$\frac{\gamma \mathcal{R}T}{(\gamma - 1) T} \frac{dT}{T} + u^2 \frac{du}{u} = \frac{Qdx}{m} \quad (2)$$

or

$$\frac{1}{(\gamma - 1) T} \frac{dT}{T} + M^2 \frac{du}{u} = \frac{Qdx}{mc^2} \quad (3)$$

But for frictionless flow in a constant, uniform duct, the continuity equation becomes

$$\frac{du}{u} + \frac{d\rho}{\rho} = 0 \quad (4)$$

and the momentum equation yields:

$$\frac{dp}{\rho} + u du = 0 \quad \text{or} \quad \frac{dp}{p} = -\gamma M^2 \frac{du}{u} \quad (5)$$

The perfect gas law yields

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (6)$$

and combining the three above equations yields

$$\frac{dT}{T} = (1 - \gamma M^2) \frac{du}{u} \quad (7)$$

and substituting this into the second equation produces

$$\frac{du}{u} = \frac{(\gamma - 1) Q}{(1 - M^2) mc^2} dx \quad (8)$$

But also since $M^2 = u^2/\gamma \mathcal{R}T$ it follows that

$$MdM = M^2 \frac{du}{u} - M^2 \frac{dT}{2T} \quad (9)$$

and using the above relations

$$\frac{dM}{dx} = \frac{M(1 + \gamma M^2)(\gamma - 1) Q}{2(1 - M^2) mc^2} \quad (10)$$

Note that dM/dx takes a different sign depending upon whether the flow is subsonic or supersonic.