Solution to Problem 290A

As described the incompressible planar potential flow past a spinning cylinder (radius $= R$) in a uniform stream of velocity, U, in the positive x direction is given by the velocity potential:

$$\phi = Ur\cos\theta + \frac{UR^2\cos\theta}{r} - \frac{\Gamma\theta}{2\pi}$$

where r, θ are polar coordinates in which $\theta = 0$ corresponds to the positive x direction and Γ is the circulation defined as positive in the clockwise direction. It follows that the velocities, u_r and u_θ in the r and θ directions, are given by

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta - \frac{UR^2\cos\theta}{r^2}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta - \frac{UR^2\sin\theta}{r^2} - \frac{\Gamma}{2\pi r}$$

Consequently the velocities on the surface of the cylinder are

$$(u_r)_{r=R} = 0$$

$$(u_\theta)_{r=R} = -2U\sin\theta - \frac{\Gamma}{2\pi R}$$

To find the pressure on the surface, $(p)_{r=R}$, we now use Bernoulli's thereom to obtain

$$(p)_{r=R} = p_\infty + \frac{\rho}{2}\left[U^2(1 - 4\sin^2\theta) - \frac{2\Gamma U}{\pi R} - \frac{\Gamma^2}{4\pi^2R^2}\right]$$

where p_∞ is the pressure in the uniform stream.

If you were to integrate this pressure over the surface in order to obtain the drag, you would find that the drag is zero as expected from D'Alembert's paradox.

Integrating the pressure over the surface in order to obtain the lift, L, per unit depth normal to the planar flow, one would evaluate the integral

$$L = \int_0^{2\pi} -(p)_{r=R} R\sin\theta d\theta$$

which leads to

$$L = \rho U \Gamma$$

This is called the Magnus effect. The lift is due to the fact that the cylinder rotation in the clockwise direction increases the surface velocity on the upper side and decreases the velocity on the lower side. By Bernoulli's equation this decreases the pressure on the upper side and increases the pressure on the lower side thus causing the positive upward lift on the cylinder. This effect also occurs with three-dimensional objects such as spheres and is particularly evident in the flight of a well-struck golf ball.