## Solution to Problem 280E

Calculate the drag on a streamlined, axisymmetric body as it moves in water.

1.) Find the form drag contribution, $C_{DF}$, to the total drag coefficient.

To find the form drag, we must examine the pressure distribution on the nose and the flat trailing portion of the body. Since we can consider the flow over the nose to be described by potential flow, $u(\theta) = \frac{3}{2}\sin\theta$, we can use Bernoulli's equation to get the corresponding pressure distribution over the surface.

$$p_\infty + \frac{1}{2}\rho U^2 = p(\theta) + \frac{1}{2}\rho u^2(\theta)$$

$$\Rightarrow p(\theta) - p_\infty = \frac{1}{2}\rho U^2\left[1 - \frac{9}{4}\sin^2\theta\right]$$

This leads to a pressure coefficient on the nose that varies with $\theta$

$$C_{p,N} = \frac{p(\theta) - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{9}{4}\sin^2\theta$$

Since the flow separates at the sharp trailing edge, we can consider the pressure to be constant on the circular base with a constant pressure coefficient.

$$c_{p,T} = -0.5$$

Our goal is the drag coefficient due to form effects, $C_{DF}$.

$$C_{DF} = \frac{D_F}{\frac{1}{2}\rho U^2\pi R^2}$$

where $D_F$ is the form drag. We calculate this drag by finding the difference between the pressure integrated over the nose and the base of the streamlined body.

$$D_F = \int p_N dA - \int p_T dA = \int (p_N - p_\infty)dA - \int (p_T - p_\infty)dA$$

Dividing this equation by $\frac{1}{2}\rho U^2\pi R^2$ gives us the form drag coefficient in terms of integrals of the pressure coefficients over the nose and tail.

$$C_{DF} = \frac{\int C_{p,N}dA}{\pi R^2} - \frac{\int C_{p,T}dA}{\pi R^2}$$

The second integral is trivial since the pressure coefficient is constant.

$$\frac{\int C_{p,T}dA}{\pi R^2} = \frac{-0.5\pi R^2}{\pi R^2} = -0.5$$

The integral over the nose is slightly more involved.

$$\begin{aligned}
\frac{\int C_{p,N}dA}{\pi R^2} &= \frac{\int\left(1 - \frac{9}{4}\sin^2\theta\right)}{\pi R^2} \\
&= \frac{1}{\pi R^2}\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left(1 - \frac{9}{4}\sin^2\theta\right)R^2\sin\theta\cos\theta d\phi d\theta \\
&= 2\int_0^{\pi/2}\left(1 - \frac{9}{4}\sin^2\theta\right)\sin\theta\cos\theta d\theta \\
&= 2\int_0^{1}\left(\sin\theta - \frac{9}{4}\sin^3\theta\right)d(\sin\theta) \\
&= 2\left[\frac{1}{2}\sin^2\theta - \frac{9}{16}\sin^4\theta\right]_0^1 = -\frac{1}{8}
\end{aligned}$$

The form drag coefficient is the difference between these integrals.

$$C_{DF} = \frac{\int C_{p,N} dA}{\pi R^2} - \frac{\int C_{p,T} dA}{\pi R^2} = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

Note: The form drag coefficient could also have been evaluated as a sigle integral, more akin to what was done in class. As was shown in class, it is sufficient to integrate over the projected area.

2.) Find the skin friction (on the cylindrical surface) contribution, $C_{DS}$, to the total drag coefficient.

Since we assume that the boundary layer remains laminar, we can use the Blasius solution to calculate the skin friction drag, $D_S$, on the cylindrical surface.

$$\begin{aligned}
D_S &= \int \tau_0 dA \\
&= \int_0^L \frac{1}{2}\rho U^2 c_f 2\pi R dx \\
&= \frac{1}{2}\rho U^2 2\pi R \int_0^L 0.664\sqrt{\frac{\nu}{Ux}} dx \\
&= \frac{1}{2}\rho U^2 2\pi R L \cdot 1.328\sqrt{\frac{\nu}{UL}}
\end{aligned}$$

This leads to a drag coefficient due to skin drag of:

$$C_{DS} = \frac{D_S}{\frac{1}{2}\rho U^2 \pi R^2} = \frac{2L}{R} \cdot 1.328\sqrt{\frac{1}{Re_L}}$$

where $Re_L = UL/\nu$ is the Reynolds number based on the length of the body. The skin friction drag coefficient can be rewritten in terms of a radius-based Reynolds number, $Re_D = U2R/\nu$ to give:

$$C_{DS} = 1.328 \cdot \frac{2L}{R}\sqrt{\frac{2R\nu}{2RUL}} = 1.328 \cdot 2\sqrt{2}\sqrt{\frac{L}{R}}\sqrt{\frac{1}{Re_R}}$$

Note: Here we used the original free stream velocity in the Blasius calculation of the shear stress. One may argue that it may be more appropriate to use the accelerated value, $\frac{3}{2}U$ for free stream. The flow after the nose can no longer be considered potenial and the higher free stream value will decrease back to U with distance along the body. Thus, there are likely regimes, based on the length of the body, in which one assumption for the velocity is preferable to the other. But, after all, we only desire an estimate of the contribution of the skin friction drag so either choice is fine.

3.) Calculate the aspect ratio, L/R, at $Re_D = 10,000$ for which the total drag is composed of equal parts form and skin friction drag.

Here we equate the two drag coefficients for the given Reynolds number.

$$C_{DF} = \frac{3}{8} = 1.328 \cdot 2\sqrt{2}\sqrt{\frac{L}{R}}\sqrt{\frac{1}{10000}} = C_{DS}$$

$$\Rightarrow \sqrt{\frac{L}{R}} = \frac{3}{8}\frac{100}{1.328 \cdot 2\sqrt{2}} = 10$$

$$\frac{L}{R} = 100$$