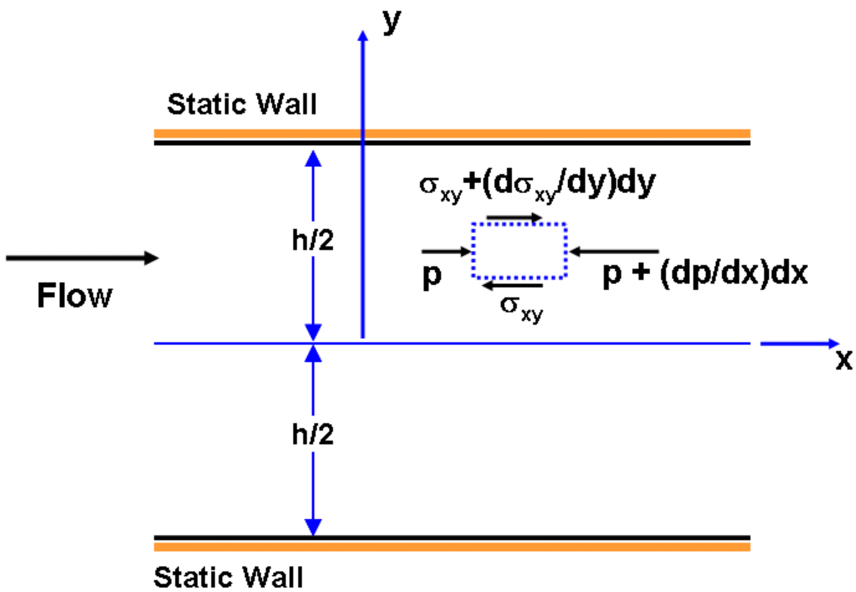
Solution to Problem 272B

To find how the shear stress, σ_{xy}, varies across planar Poiseuille flow, consider the small control volume, $dx \times dy$ shown by the blue dotted lines in the following diagram: Noting the pressures acting on the vertical sides and the shear stresses acting



on the horizontal sides and recognizing that in steady, fully developed flow the fluid velocities will not change with time or in the x direction, it follows that the forces due to these pressures and shear stresses must balance so that

$$p\ dy - \left(p + \frac{\partial p}{\partial x}dx\right)dy - \sigma_{xy}\ dx + \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y}dy\right)dx = 0$$

so that

$$\frac{\partial \sigma_{xy}}{\partial y} = -\frac{\partial p}{\partial x} = -\frac{dp}{dx}$$

since p does not vary with y. This can be integrated with respect to y to yield

$$\sigma_{xy} = -\frac{dp}{dx}y + C$$

where C is an integration constant and by symmetry about the x axis we must have $C = 0$ so that

$$\sigma_{xy} = -\frac{dp}{dx}y$$

Thus σ_{xy} varies linearly with y and $A = -dp/dx$.

Given the stated assumptions when the core is turbulent we may write the shear stress as

$$\sigma_{xy} = \rho\ell^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \rho\frac{h^2}{16}\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

where ℓ is the mixing length so that, with the result above,

$$-\frac{dp}{dx}y = \rho\frac{h^2}{16}\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

so that taking the square root and choosing the appropriate sign

$$\frac{\partial\bar{u}}{\partial y} = -\left(-\frac{1}{\rho}\frac{dp}{dx}\right)^{\frac{1}{2}}\frac{4y^{\frac{1}{2}}}{h}$$

Note that the choice of sign is determined by the fact that $(-dp/dx)$ is positive since the flow proceeds in the positive x direction and the fact that for $y > 0$ we must have a negative $\partial\bar{u}/\partial y$. Integrating this equation to obtain $\bar{u}(y)$:

$$\bar{u} = -\frac{8}{3h}\left(-\frac{1}{\rho}\frac{dp}{dx}\right)^{\frac{1}{2}} y^{\frac{3}{2}} + \text{constant}$$

where the constant is determined by the condition that $\bar{u} = 0$ on $y = h/2$ so that, finally, for $y > 0$

$$\bar{u} = \frac{4}{3}\left(-\frac{1}{\rho}\frac{dp}{dx}\right)^{\frac{1}{2}}\left(\frac{h}{2}\right)^{\frac{1}{2}}\left[1 - \left(\frac{2y}{h}\right)^{\frac{3}{2}}\right]$$

Note that it also follows by symmetry that for $y < 0$

$$\bar{u} = \frac{4}{3}\left(-\frac{1}{\rho}\frac{dp}{dx}\right)^{\frac{1}{2}}\left(\frac{h}{2}\right)^{\frac{1}{2}}\left[1 - \left(\frac{-2y}{h}\right)^{\frac{3}{2}}\right]$$