Solution to Problem 271A

The velocity profile for a turbulent boundary layer of incompressible fluid on a flat plate (where U = constant) is approximated as

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

Finding α, an expression for δ.

$$\begin{aligned}
\alpha = \frac{\delta_M}{\delta} &= \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)\ \mathrm{d}\left(\frac{y}{\delta}\right) \\
&= \int_0^1 \left[\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right] \mathrm{d}\left(\frac{y}{\delta}\right) \\
&= \left[\frac{7}{8}\left(\frac{y}{\delta}\right)^{\frac{8}{7}} - \frac{7}{9}\left(\frac{y}{\delta}\right)^{\frac{9}{7}}\right|_0^1 \\
&= \frac{7}{8} - 79 \\
&= \frac{7}{72} = 0.0972
\end{aligned}$$

From the K.M.I.E.,

$$\begin{aligned}
\tau_W &= \rho\frac{\mathrm{d}\left(U^2\delta_M\right)}{\mathrm{d}x} + \rho\delta_D U\frac{\mathrm{d}U}{\mathrm{d}x} \\
&= \rho U^2\frac{\mathrm{d}\left(\alpha\delta\right)}{\mathrm{d}x} \\
&= \rho U^2\alpha\frac{\mathrm{d}\delta}{\mathrm{d}x}
\end{aligned}$$

where α and U are constants. If the wall shear stress for this turbulent profile is assumed to be given by $\tau_W = 0.023\rho U^2\left(\nu/\delta U\right)^{\frac{1}{4}}$,

$$\begin{aligned}
\tau_W = \rho U^2\alpha\frac{\mathrm{d}\delta}{\mathrm{d}x} &= 0.023\rho U^2\left(\nu/\delta U\right)^{\frac{1}{4}} \\
\delta^{\frac{1}{4}}\mathrm{d}\delta &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\mathrm{d}x \\
\frac{4}{5}\delta^{\frac{5}{4}} &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x + c
\end{aligned}$$

To evaluate c, evaluate expression at $x = x_0$, $\delta = \delta_0$:

$$c = \frac{4}{5}\delta_0^{\frac{5}{4}} - \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x_0$$

$$\begin{aligned}
\frac{4}{5}\left(\delta^{\frac{5}{4}} - \delta_0^{\frac{5}{4}}\right) &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0) \\
\delta^{\frac{5}{4}} &= \frac{5}{4}\left(\frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0) + \frac{4}{5}\delta_0^{\frac{5}{4}}\right) \\
\delta &= \left[\frac{5}{4}\left(\frac{0.023}{\alpha}\right)\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0) + \delta_0^{\frac{5}{4}}\right]^{\frac{4}{5}} \\
&= \left[0.296\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0) + \delta_0^{\frac{5}{4}}\right]^{\frac{4}{5}}
\end{aligned}$$

This solution will be valid for $x \geq x_0$, i.e., within the turbulent boundary layer.