Solution to Problem 269A

[1] For the hypothetical laminar sub-layer

$$y* = \frac{\delta_{LSL} u_\tau}{\nu} = 5 \qquad ; \qquad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$$

and therefore for the roughness, ϵ, to extend to the same height as the sublayer requires

$$\epsilon = \frac{5\nu}{u_\tau}$$

[2] If $\epsilon > \frac{5\nu}{u_\tau}$ then there is no sublayer and the flow is entirely dominated by turbulence and by Reynolds' shear stresses, $-\rho\overline{u'v'}$. Note that it is independent of the viscosity, ν. Hence the only length scale left is the roughness height, ϵ while the velocity must still scale with u_τ. Therefore by dimensional analysis we must have

$$\frac{u}{u_\tau} = f\left(\frac{y}{\epsilon}\right)$$

[3] Assuming $\tau = -\rho\overline{u'v'}$ to be constant and equal to τ_w it follows from Prandtl's mixing length hypothesis, namely,

$$-\overline{u'v'}\frac{u}{u_\tau} = \kappa^2 y^2 \left(\frac{d\overline{u}}{dy}\right)^2$$

that

$$\kappa y \left(\frac{d\overline{u}}{dy}\right) = u_\tau$$

Integrating this with respect to y:

$$\frac{\overline{u}(y)}{u_\tau} = \frac{1}{\kappa}\ln(y) + C$$

where C is an integration constant which needs to be determined by experiment.