Solution to Problem 255A

Thwaites' method for the prediction of laminar boundary layer separation is based on the parameter, λ, defined as:

$$\lambda = \frac{\delta_m^2}{\nu}\frac{dU_\infty}{dx}$$

$$\delta_m^2 = (\delta_m)^2_{x=0} + \frac{0.45\nu}{U_\infty^6}\int_0^x U_\infty^5 dx$$

where δ_m is the momentum thickness of the boundary layer, $(\delta_m)_{x=0}$ is the momentum thickness at $x = 0$, U_∞ is the velocity just outside the boundary layer, ν is the kinematic viscosity and x is the streamwise surface coordinate. From potential flow over a cylinder:

$$U_\infty = 2U\sin\theta = 2U\sin\frac{x}{R}$$

Proceeding to calculate λ for this case:

$$\frac{dU_\infty}{dx} = \frac{2U}{R}\cos\frac{x}{R} = \frac{2U}{R}\cos\theta$$

and using the input that the momentum thickness at the front stagnation point is zero, $\delta_m|_{x=0}$ it follows that

$$\begin{aligned}
\delta_m^2 &= \frac{0.45\nu}{U_\infty^6}\int_0^x U_\infty^5 dx \\
&= \frac{0.45\nu}{(2U\sin^6\theta)}\int_0^\theta (2U\sin^5\theta)Rd\theta \\
&= \frac{0.45\nu R}{2U\sin^6\theta}\int_0^\theta \sin\theta(1-\cos^2\theta)d\theta \\
&= \frac{0.45\nu R}{2U\sin^6\theta}\int_0^\theta (-1+2\cos^2\theta-\cos^4\theta)d(\cos\theta) \\
&= \frac{0.45\nu R}{2U\sin^6\theta}\left[-\cos\theta+\frac{2}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right]_0^\theta \\
&= \frac{0.45\nu R}{2U(1-\cos^2\theta)^3}\left[\frac{8}{15}-\cos\theta+\frac{2}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right]
\end{aligned}$$

and therefore

$$\lambda = \frac{0.45\cos\theta}{(1-\cos\theta^2)^3}\left[\frac{8}{15}-\cos\theta+\frac{2}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right]$$

Thwaites' criterion is that separation occurs at $\lambda = -0.09$ and therefore the equation that determines the point of separation becomes

$$0.45\left(\frac{8}{15}y - y^2 + \frac{2}{3}y^4 - \frac{1}{5}y^6\right) = -0.09\left(1-3y^2+3y^4-y^6\right)$$

where, for convenience, we have used $y = \cos\theta$. Solving for y by iteration or other methods we find

$$y = -0.2268$$

and therefore separation is predicted to occur at

$$\theta_{separation} = 103^\circ$$

Experimentally separation is observed to occur about 84°, considerably earlier. To get closer to this one would need to calculate a more accurate potential flow in which the flow separates at, say, 103° and then recalculate the separation point. In other words one would need to iterate toward a more realistic solution using both a potential flow calculation and a boundary layer calculation.