Solution to Problem 250Z

To solve this problem we employ approximate boundary layer methods based on the Karman Momentum Integral Equation (KMIE):

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\delta_M\right) + \delta_D U \frac{dU}{dx}$$

where τ_w is the wall shear stress, U is the velocity exterior to the boundary layer, δ is the boundary layer thickness, ν and ρ are the kinematic viscosity and density of the fluid, x is the streamwise distance along the wall surface and α, β and γ are the usual profile parameters given by

$$\begin{aligned}\alpha &= \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 (3\eta - 3\eta^2 + \eta^3)(1 - 3\eta + 3\eta^2 - \eta^3) d\eta \\ &= 0.107\end{aligned}$$

$$\beta = \left(\frac{d(u/U)}{d(y/\delta)}\right)_{y=0} = \left(\frac{d}{d\eta}(3\eta - 3\eta^2 + \eta^3)\right)_{y=0} = 3$$

$$\begin{aligned}\gamma &= \int_0^1 \left(1-\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 (1 - 3\eta + 3\eta^2 - \eta^3) d\eta \\ &= 0.25\end{aligned}$$

Then the KMIE becomes

$$\frac{\nu\beta}{\alpha}\frac{1}{\delta} = U\frac{d\delta}{dx} + \left(2+\frac{\gamma}{\alpha}\right)\delta\frac{dU}{dx}$$

Then if $U = Cx^{\frac{1}{9}}$ and $\delta = Ax^k$ it follows that

$$\frac{\nu\beta}{C\alpha A^2} = kx^{2k-\frac{8}{9}} + \frac{1}{9}\left(2+\frac{\gamma}{\alpha}\right)x^{2k-\frac{8}{9}}$$

and therefore

$$k = \frac{4}{9}$$

and

$$A = \left[\frac{9\nu\beta}{C(6\alpha+\gamma)}\right]^{\frac{1}{2}}$$