Solution to Problem 250C

To solve this problem we employ approximate boundary layer methods based on the von Karman momentum integral equation:

$$\frac{\nu U \beta}{\delta} = \frac{d}{dx}(\alpha \delta U^2) + \gamma \delta U \frac{dU}{dx}$$

where $U(x)$ is the velocity outside the boundary layer, δ is a measure of the boundary layer thickness, ν is the kinematic viscosity of the fluid and α, β and γ are the profile parameters. Using $\eta = y/\delta$ for convenience, the profile parameters for the given velocity profile can be calculated as:

$$\begin{aligned} \alpha &= \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 (3\eta - 3\eta^2 + \eta^3)(1 - 3\eta + 3\eta^2 - \eta^3) d\eta \\ &= 0.107 \end{aligned}$$

$$\beta = \left(\frac{d(u/U)}{d(y/\delta)}\right)_{y=0} = \left(\frac{d}{d\eta}(3\eta - 3\eta^2 + \eta^3)\right)_{y=0} = 3$$

$$\begin{aligned} \gamma &= \int_0^1 \left(1-\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 (1 - 3\eta + 3\eta^2 - \eta^3) d\eta \\ &= 0.25 \end{aligned}$$

In this case the Karman momentum integral equation thus becomes

$$\frac{\nu\beta}{\alpha}\frac{1}{\delta} = U\frac{d\delta}{dx} + \left(2 + \frac{\gamma}{\alpha}\right)\delta\frac{dU}{dx}$$

Then if $U = Cx^{\frac{1}{9}}$ and $\delta = Ax^k$ it follows that

$$\frac{\nu\beta}{C\alpha A^2} = kx^{2k-\frac{8}{9}} + \frac{1}{9}\left(2+\frac{\gamma}{\alpha}\right)x^{2k-\frac{8}{9}}$$

and therefore

$$k = \frac{4}{9}$$

and

$$A = \left[\frac{9\nu\beta}{C(6\alpha+\gamma)}\right]^{\frac{1}{2}}$$