Solution to Problem 250A

It is assumed that the velocity profile of a laminar boundary layer on a flat plate with zero pressure gradient is approximately of the form:

$$\frac{u(y)}{U} = sin\left(\frac{\pi y}{2\delta}\right)$$

for $0 < y < \delta$ and $u(y)/U = 1$ for $y > \delta$. This profile satifies the boundary conditions that (1) $u(0) = 0$ (2) $u(\delta) = U$ and (3) du/dy at $y = \delta$ is zero. We are asked to find the displacement thickness δ_D, the momentum thickness δ_M and the skin friction drag D on the plate with length L and breadth B.

For convenience we introduce the dimensionless coordinate $\eta = y/\delta$ so that:

$$\frac{u(\eta)}{U} = sin\left(\frac{\pi\eta}{2}\right)$$

First we calculate the profile parameters for this self-similar velocity profile:

$$\gamma = \frac{\delta_D}{\delta} = \int_0^1 \left(1 - \frac{u(\eta)}{U}\right) = 1 - \frac{2}{\pi} = 0.3634$$

$$\alpha = \frac{\delta_M}{\delta} = \int_0^1 \frac{u(\eta)}{U}\left(1 - \frac{u(\eta)}{U}\right) = \frac{2}{\pi} - \frac{1}{2} = 0.1366$$

and

$$\beta = \frac{\partial \frac{u(y)}{U}}{\partial \frac{y}{\delta}}|_{y/\delta=0} = \frac{d\frac{u(y)}{U}}{d\eta}|_{\eta=0} = \frac{\pi}{2} = 1.571$$

Then we use the von Karman boundary layer momentum integral to obtain

$$\delta = \left(\frac{2\beta}{\alpha}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2} = 4.796\left(\frac{\nu x}{U}\right)^{1/2}$$

The displacement thickness is given by:

$$\delta_D = \gamma\delta = 1.743\left(\frac{\nu x}{U}\right)^{1/2}$$

which we compare with the value of δ_D from the Blasius flat plate boundary layer solution namely

$$\delta_D = 1.72\left(\frac{\nu x}{U}\right)^{1/2}$$

The momentum thickness for the approximate profile is:

$$\delta_M = \alpha\delta = 0.655\left(\frac{\nu x}{U}\right)^{1/2}$$

in comparison with the Blasius solution result namely:

$$\delta_M = 0.664\left(\frac{\nu x}{U}\right)^{1/2}$$

To obtain the skin-friction drag we first compute the wall shear stress, τ_w:

$$\tau_w = \mu \left.\frac{\partial u(y)}{\partial y}\right|_{\text{wall}}$$

which, for the approximate profile, leads to

$$\tau_w = \frac{\rho\nu U\pi}{2\delta} = 0.3275\rho U^{3/2}\left(\frac{\nu}{x}\right)^{1/2}$$

and therefore the drag, D, on one side of the plate is given by:

$$D = \int_0^L \tau_w dx = 0.655 \rho U^{3/2} (\nu L)^{1/2}$$

This compares with the Blasius result namely:

$$D = 0.664 \rho U^{3/2} (\nu L)^{1/2}$$

Consequently this approximate sinusoidal velocity profile yields a fair approximate to the results for the exact Blasius profile.