

### Solution to Problem 241C

The actual cross-section of the duct as a function of  $x$  is

$$b [h_0 + Hx^k]$$

but due to the displacement thickness of the boundary layers on the walls,  $\delta_D(x)$ , the effective cross-sectional area of the duct is

$$bh_0 + bHx^k - 2b\delta_D$$

Using the Falkner-Skan solutions for  $U = Cx^m$  we find that the displacement thickness

$$\delta_D = \left\{ \frac{4\nu x^{(1-m)}}{C} \right\}^{\frac{1}{2}} \int_0^\infty \left\{ 1 - \frac{dF}{d\eta} \right\} d\eta$$

For convenience we denote the number that the integral represents by

$$I(m) = \int_0^\infty \left\{ 1 - \frac{dF}{d\eta} \right\} d\eta$$

where, since  $F(\eta)$  depends on  $m$ , the value of  $I(m)$  changes with  $m$ .

**Part 1:** If the longitudinal pressure gradient is to be zero, we must have  $U$  constant and therefore  $m = 0$  so that the appropriate boundary layer solution is the Blasius laminar boundary layer solution for which

$$\delta_D = 1.72 \left\{ \frac{\nu x}{U} \right\}^{\frac{1}{2}}$$

But conservation of mass requires that  $U$  multiplied by the effective cross-sectional area must be a constant and since  $U$  is a constant in this case, it follows that the effective cross-sectional area must be a constant. The only way that this can be true is if

$$Hx^k = 2\delta_D$$

and it therefore follows that, in this case,

$$k = 1/2 \quad \text{and} \quad H = 3.44(\nu/U)^{\frac{1}{2}}$$

**Part 2:** In order to have  $dU/dx = C$  we must have  $U = Cx$  in the Falkner-Skan solution. Therefore, in this case,  $m = 1$  and hence

$$\delta_D = \text{constant} = 2I(1)(\nu/C)^{\frac{1}{2}}$$

But mass conservation requires that  $U$  multiplied by the effective cross-sectional area must be a constant and therefore

$$\text{Effective Cross-sectional Area} = \frac{\text{constant}}{Cx} = b(h_0 + Hx^k - 2\delta_D)$$

This can only be true if

$$k = -1 \quad \text{and} \quad h_0 = 2\delta_D = 4I(1)(\nu/C)^{\frac{1}{2}}$$