Solution to Problem 160E:

In class we derived a relation between the shear stress and the pressure gradient for fully developed pipe flow that was good for any kind of fluid. I repeat that derivation here for the sake of completeness:

Consider a cylindrical control volume within the pipe of axial length dx and extending out to some arbitrary radius r within the pipe. Since the velocities at the two ends of this control volume are identical there is no net momentum flux out of this control volume. Consequently the forces acting on this control volume must balance. If the pressure at the inlet end is denoted by p the force in the x direction due to that pressure is $p\pi r^2$ in the positive direction. Then the pressure on the outlet end will be $(p+(dp/dx)dx)$ and this creates a force $(p+(dp/dx)dx)\pi r^2$ in the negative x direction. Thirdly, if the shear stress at radius r is denoted by σ this will impose a force equal to $2\pi r\sigma \; dx$ in the negative x direction. Equating these forces leads to the result:

$$\sigma(r) \;=\; \frac{r}{2}\left(-\frac{dp}{dx}\right) \tag{1}$$

Therefore in the present problem

$$\sigma(r) \;=\; \frac{r}{2}\left(-\frac{dp}{dx}\right) \;=\; C\left(-\frac{du}{dr}\right)^4 \tag{2}$$

where, since $(-dp/dx)$ is independent of r, it can be regarded as a constant. Then

$$\left(-\frac{du}{dr}\right) \;=\; \frac{r^{1/4}}{(2C)^{1/4}}\left(-\frac{dp}{dx}\right)^{1/4} \tag{3}$$

and integrating

$$u \;=\; -\frac{4r^{5/4}}{5(2C)^{1/4}}\left(-\frac{dp}{dx}\right)^{1/4} + \text{constant} \tag{4}$$

where the integration constant is determined by the condition that $u=0$ at $r=R$ so that

$$u \;=\; \frac{4}{5(2C)^{1/4}}\left(-\frac{dp}{dx}\right)^{1/4}\left\{R^{5/4}-r^{5/4}\right\} \tag{5}$$

To find the average volumetric velocity, $\overline{u}$, we integrate over the cross-section of the pipe and divide by the area, πR^2:

$$\overline{u} \;=\; \frac{1}{\pi R^2}\int_0^R 2\pi r u \; dr \;=\; \frac{4R^{5/4}}{13(2C)^{1/4}}\left(-\frac{dp}{dx}\right)^{1/4} \tag{6}$$

and it then follows that the friction factor, f, is given by

$$f \;=\; \frac{1}{\pi R^2}\int_0^R 2\pi r u \; dr \;=\; \frac{169(2C)^{1/2}}{4\rho R^{3/2}}\left(-\frac{dp}{dx}\right)^{1/2} \tag{7}$$