Solution to Problem 150N:

The constitutive laws for an incompressible, Newtonian fluid (dynamic viscosity, μ) when written in spherical coordinates, (r, θ, ϕ), with velocities u_r, u_θ, u_ϕ in the r, θ, ϕ directions become:

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} \tag{1}$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \tag{2}$$

$$\sigma_{\phi\phi} = -p + 2\mu \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} \right) \tag{3}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu \left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right) \tag{4}$$

$$\sigma_{r\phi} = \sigma_{\phi r} = \mu \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) \right) \tag{5}$$

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu \left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) \right) \tag{6}$$

Since the flow is purely radial ($u_r \neq 0$, $u_\theta = 0$ and $u_\phi = 0$), the continuity equation for an incompressible fluid requires that

$$\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0 \tag{7}$$

and therefore

$$r^2 u_r = f(t) \tag{8}$$

or some function, f, of t. But since $u_r = dR/dt$ at $r = R(t)$:

$$f(t) = R^2 \frac{dR}{dt} \quad \text{and} \quad u_r = \frac{R^2}{r^2}\frac{dR}{dt} \tag{9}$$

Also, setting $u_r \neq 0$, $u_\theta = 0$ and $u_\phi = 0$, the stresses become

$$\sigma_{rr} = -p - \frac{4\mu R^2}{r^3}\frac{dR}{dt} \quad ; \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p + \frac{2\mu R^2}{r^3}\frac{dR}{dt} \tag{10}$$

$$\sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{r\phi} = 0 \tag{11}$$

But the balance of forces on a thin lamina of the bubble surface requires that

$$p_G = -\left(\sigma_{rr}\right)_{r=R} + \frac{2S}{R} \tag{12}$$

where p_G is the gas pressure inside the bubble.

Therefore the answer to the question (using $dR/dt = V$) is

$$p_G = p + \frac{4\mu V}{R} + \frac{2S}{R} \tag{13}$$

where p is the pressure in the liquid at the bubble surface.

Note that in the liquid at the bubble surface, p is equal to the mean of three normal stresses.