Solution to Problem 150J:

Since the flow is purely radial (in spherical coordinates, $u_r \neq 0$, $u_\theta = u_\phi = 0$, the continuity equation for an incompressible fluid requires that

$$\frac{\partial}{\partial r}\left(r^2 u_r\right) \;=\; 0 \tag{1}$$

and therefore

$$r^2 u_r \;=\; f(t) \tag{2}$$

or some function, f, of t. But since $u_r \;=\; dR/dt$ at $r = R(t)$:

$$f(t) \;=\; R^2\frac{dR}{dt} \quad \text{and} \quad u_r \;=\; \frac{R^2}{r^2}\frac{dR}{dt} \tag{3}$$

For this radial flow Euler's equations in the θ and ϕ directions are automatically satisfied and the equation in the r direction reduces to

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r}\right) \;=\; -\frac{\partial p}{\partial r} \tag{4}$$

and substituting the above expression for u_r:

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} \;=\; \frac{2R}{r^2}\left(\frac{dR}{dt}\right)^2 \;+\; \frac{R^2}{r^2}\frac{d^2R}{dt^2} \;-\; \frac{2R^4}{r^5}\left(\frac{dR}{dt}\right)^2 \tag{5}$$

Integrating

$$\frac{p(r,t)}{\rho} \;=\; \frac{2R}{r}\left(\frac{dR}{dt}\right)^2 \;+\; \frac{R^2}{r}\frac{d^2R}{dt^2} \;-\; \frac{R^4}{2r^4}\left(\frac{dR}{dt}\right)^2 \;+\; C \tag{6}$$

where C is an integration constant that follows when the condition that $p \to p_\infty$ as $r \to \infty$ is applied. Also in the absence of surface tension $p(R,t)$ is equal to the pressure in the bubble, p_B, so that

$$\frac{(p_B - p_\infty)}{\rho} \;=\; \frac{3}{2}\left(\frac{dR}{dt}\right)^2 \;+\; R\frac{d^2R}{dt^2} \tag{7}$$

which is the Rayleigh equation for bubble growth.