Solution to Problem 150I:

In cylindrical coordinates, (r, θ, z), the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, μ, and density, ρ, are

$$\rho\left[\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right] = -\frac{\partial p}{\partial r} + f_r + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] \tag{1}$$

$$\rho\left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] \tag{2}$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + f_z + \mu\nabla^2 u_z \tag{3}$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinate directions, p is the pressure, f_r, f_θ, f_z are the body force components in the r, θ, z directions and the operators D/Dt and ∇^2 are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \tag{4}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{5}$$

Then since the flow is steady, neglecting the convective inertial terms and recognizing that the body forces will not contribute to the drag so we will neglect those the equations of motion become

$$0 \;=\; -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] \tag{6}$$

$$0 \;=\; -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] \tag{7}$$

along with $\partial p/\partial z = 0$.

With the given expressions for ψ, u_r and u_θ it follows that

$$\frac{\partial u_r}{\partial r} \;=\; \frac{2UR^2}{r^3}\cos\theta \quad ; \quad \frac{\partial^2 u_r}{\partial r^2} \;=\; -\frac{6UR^2}{r^4}\cos\theta \quad ; \quad \frac{\partial^2 u_r}{\partial \theta^2} \;=\; -U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{8}$$

$$\frac{\partial u_\theta}{\partial r} \;=\; \frac{2UR^2}{r^3}\sin\theta \quad ; \quad \frac{\partial^2 u_\theta}{\partial r^2} \;=\; -\frac{6UR^2}{r^4}\sin\theta \quad ; \quad \frac{\partial u_r}{\partial \theta} \;=\; -U\left(1 - \frac{R^2}{r^2}\right)\sin\theta \tag{9}$$

and substituting into the above equations of motion, it transpires that

$$\frac{\partial p}{\partial r} \;=\; \frac{\partial p}{\partial \theta} \;=\; 0 \tag{10}$$

and therefore p is a simple constant. Moreover

$$(\sigma_{rr})_{r=R} \;=\; \frac{4\mu U}{R}\cos\theta \quad \text{and} \quad (\sigma_{r\theta})_{r=R} \;=\; \frac{4\mu U}{R}\sin\theta \tag{11}$$

Now the drag D per unit depth normal to the flow will be given by

$$D \;=\; -2\int_0^\pi (\sigma_{rr})_{r=R}\; R\cos\theta\; d\theta \;+\; 2\int_0^\pi (\sigma_{r\theta})_{r=R}\; R\sin\theta\; d\theta \tag{12}$$

and substituting for the stresses this yields **D=0**.