Solution to Problem 150E

a.) Since the flow is steady, planar, and incompressible the continuity equations is:

$$\frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} = 0$$

Since the flow is the same at all locations, s, along the wall it follows that $\partial u_s/\partial s = 0$ and therefore from this continuity equation it follows that $\partial u_n/\partial n = 0$. Consequently u_n is a constant independent of n. But since $u_n = 0$ at the wall it must be zero everywhere.

The Navier-Stokes equation in the s-direction becomes

$$\rho\left[\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} + w\frac{\partial u_s}{\partial z}\right] = -\frac{\partial p}{\partial s} + \rho g\sin\theta + \mu\left[\frac{\partial^2 u_s}{\partial s^2} + \frac{\partial^2 u_s}{\partial n^2} + \frac{\partial^2 u_s}{\partial z^2}\right]$$

Since the flow is the same at all s locations and since the pressure on the free surface is the same at all s locations it follows that $\partial p/\partial s = 0$. In addition since the flow is steady, planar, $u_n = 0$, and $\partial u_s/\partial s = 0$, it follows that:

$$\frac{d^2 u_s}{dn^2} = -\frac{\rho g}{\mu}\sin\theta$$

Integrating twice with respect to n:

$$u_s(n) = -\frac{\rho g}{2\mu}\sin\theta n^2 + c_1 n + c_2$$

The boundary conditions are the no slip condition at the plate and no shear stress at the free surface. These yield

$$u(0) = 0 = c_2$$

$$\tau(h) = \mu\left.\frac{\partial u}{\partial n}\right|_{n=h} = \mu\left(-\frac{\rho g}{\mu}h\sin\theta + c_1\right) = 0$$

and therefore

$$c_1 = \frac{\rho g h}{\mu}\sin\theta$$

and

$$u(n) = \frac{\rho g}{2\mu}\sin\theta\ \ n(2h-n)$$

Therefore

$$C = \frac{\rho g}{2\mu}\sin\theta$$

b.) To find the pressure acting on the plate if the atmospheric pressure is denoted by p_a we write the Navier-Stokes equation in the n-direction:

$$\rho\left[\frac{\partial u_n}{\partial t} + u_s\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial n} + w\frac{\partial u_n}{\partial z}\right] = -\frac{\partial p}{\partial n} - \rho g\cos\theta + \mu\left[\frac{\partial^2 u_n}{\partial s^2} + \frac{\partial^2 u_n}{\partial n^2} + \frac{\partial^2 u_n}{\partial z^2}\right]$$

Since the flow is steady, planar, and $u_n = 0$, this becomes:

$$\frac{\partial p}{\partial n} = -\rho g\cos\theta$$

Therefore

$$p(n) = -\rho g n\cos\theta + c_3$$

At the surface the pressure is equal to atmospheric pressure:

$$p(h) = -\rho g h\cos\theta + c_3 = p_A$$

and therefore

$$c_3 = p_A + \rho g h \cos\theta$$

Thus

$$p(y) = p_A + \rho g \cos\theta (h - y)$$

and pressure acting on the plate is then:

$$p(0) = p_A + \rho g h \cos\theta$$