Solution to Problem 150A

1.) Since the flow is steady, planar, and incompressible the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocity in the vertical direction, v, is zero at both boundaries and thus everywhere in the flow, so the continuity equation dictates that:

$$\frac{\partial u}{\partial x} = 0$$

so u is only a function of y, $u = u(y)$.

The Navier-Stokes equation in the y-direction reduces to

$$\frac{\partial p}{\partial y} = 0$$

and therefore the pressure can only be a function of x.

The Navier-Stokes equation in the **x-direction** is:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Since the flow is steady, planar, $v = 0$, and $\frac{\partial u}{\partial x} = 0$, this becomes:

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice with respect to y and noting that $\partial p/\partial x$ is a simple constant for this operation:

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$

We now use the boundary conditions to evaluate the constants c_1, c_2 :

$$u(0) = c_2 = 0$$

$$u(H) = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + c_1 H = U$$

Therefore

$$c_1 = \frac{U}{H} - \frac{H}{2\mu}\frac{\partial p}{\partial x}$$

Inserting these values for the constants, the velocity distribution is:

$$\frac{u(y)}{U} = \frac{y}{H} - \frac{H^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

2.) Find the magnitude and direction of the particular pressure gradient for which there would be zero net volume flow in the x direction. Evaluating the volume flow rate, Q, per unit depth normal to the sketch:

$$\begin{aligned} Q &= \int_0^H u(y)dy \\ &= \int_0^H \left\{U\frac{y}{H} + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - Hy\right)\right\}dy \\ &= \frac{1}{2}UH - \frac{1}{12\mu}\frac{\partial p}{\partial x}H^3 \end{aligned}$$

Therefore the particular pressure gradient, $\frac{\hat{\partial p}}{\partial x}$, for which there will be no net volume flow ($Q = 0$) will be:

$$\frac{\hat{\partial p}}{\partial x} = \frac{6\mu U}{H^2}$$

The pressure gradient is positive, so the pressure will need to increase in the positive x-direction to offset the effect of the moving upper plate.