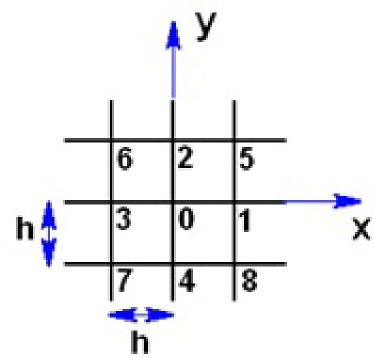
Solution to Problem 138B:



Since by Taylor's expansion:

$$\phi_5 \; = \; \phi_2 \; + \; h \left(\frac{\partial \phi}{\partial x} \right)_2 + \frac{h^2}{2} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_2 \; + \; \ldots$$

and

$$\phi_2 \; = \; \phi_0 \; + \; h \left(\frac{\partial \phi}{\partial y} \right)_0 \; + \; \frac{h^2}{2} \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \; + \; \ldots$$

and

$$\left(\frac{\partial \phi}{\partial x} \right)_2 \; = \; \left(\frac{\partial \phi}{\partial x} \right)_0 \; + \; h \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0 \; + \; \ldots$$

and

$$\left(\frac{\partial^2 \phi}{\partial x^2} \right)_2 \; = \; \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 \; + \; h \left(\frac{\partial^3 \phi}{\partial x^2 \partial y} \right)_0 \; + \; \ldots$$

Substituting the last three expressions into the first yields

$$\phi_5 \; = \; \phi_0 \; + \; h \left(\frac{\partial \phi}{\partial y} \right)_0 \; + \; h \left(\frac{\partial \phi}{\partial x} \right)_0 \; + \; \frac{h^2}{2} \left\{ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0 \right\} \; + \; O(h^3)$$

Similarly

$$\phi_7 \; = \; \phi_0 \; - \; h \left(\frac{\partial \phi}{\partial y} \right)_0 \; - \; h \left(\frac{\partial \phi}{\partial x} \right)_0 \; + \; \frac{h^2}{2} \left\{ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0 \right\} \; - \; O(h^3)$$

Therefore

$$\phi_5 \; + \; \phi_7 \; = \; 2\phi_0 \; + \; h^2 \left\{ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0 \right\} \; + \; O(h^4)$$

Similarly

$$\phi_6 \; + \; \phi_8 \; = \; 2\phi_0 \; + \; h^2 \left\{ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 - 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0 \right\} \; + \; O(h^4)$$

Therefore

$$\phi_5 \; + \; \phi_7 \; - \; \phi_6 - \; \phi_8 \; \approx \; 4h^2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_0$$

and

$$\left(\frac{\partial^2\phi}{\partial x \partial y}\right)_0 \approx \frac{\phi_5 + \phi_7 - \phi_6 - \phi_8}{4h^2}$$