Solution to Problem 130A

The solutions of the potential flows associated with wave propagation must, of course, satisfy Laplace's equation:

$$\nabla^2\phi = 0$$

The wave of interest is a traveling wave (as opposed to a standing wave). A progressive wave traveling to the right with velocity c can be described by the velocity potential:

$$\phi = \left(Ae^{ny} + Be^{-ny}\right) cos\left(n(x - ct)\right)$$

with wave number $n = 2\pi/\lambda$. It is readily shown that this satifies Laplace's equation. The two constants, A and B, in this equation and the unknown wave velocity c remain to be determined using the boundary conditions.

The appropriate boundary condition at the channel bottom is that the normal velocity, the velocity in the y-direction, must be zero:

$$v = \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0$$

The kinematic condition at the water surface requires that

$$v = \frac{\partial\eta}{\partial t}$$

When the amplitude of the wave, η, is small such that $\eta << h$ the derivative may be approximately evaluated at $y = h$. The dynamic boundary condition follows from the unsteady Bernoulli equation and the condition that the pressure on the free surface is constant:

$$\frac{\partial\phi}{\partial t}|_{y=h} + g(h + \eta) = constant$$

where the dynamic contribution (the $0.5\rho|u|^2$ term) is a second order effect when the amplitude of the wave is small relative to the wavelength. Again, the derivative in this equation may be approximately evaluated at $y = h$ when the amplitude of the wave, η, is small such that $\eta << h$.

Using the boundary condition at the bottom, it follows that

$$A = B$$

and hence

$$\phi = A\left(e^{ny} - e^{-ny}\right)\cos n(x - ct) = A\cosh y \cos n(x - ct)$$

Differentiating this to find v, using the kinematic condition at the water surface to find η and then substituting into the dynamic condition at the free surface one obtains

$$-An^2c^2\cosh nh + gAn\sinh nh = 0$$

or

$$c = \sqrt{\frac{g}{n}\tanh nh}$$

Substituting $n = 2\pi/\lambda$, the speed of propagation is obtained as:

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}}$$