Solution to Problem 123A

From the previous problem, the velocity potential for the flow past a sphere is

$$\phi(r,\theta) = U\cos\theta\left(r + \frac{1}{2}\frac{R^3}{r^2}\right)$$

The tangential velocity component on the surface of the sphere is:

$$u_\theta(R,\theta) = -U\sin\theta\left(1 + \frac{1}{2}\frac{R^3}{r^3}\right)\Bigg|_{r=R} = -\frac{3}{2}U\sin\theta$$

Using Bernoulli's equation, we find the relation between the pressure at infinity, P_∞, and the pressure, P_s, on the surface at a point where the angle from the front stagnation point is θ is

$$P_\infty + \frac{1}{2}\rho U^2 = P_s + \frac{1}{2}\rho\left(-\frac{3}{2}U\sin\theta\right)^2$$
$$P_\infty - P_s = -\frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$$

If the pressures match at the equator $\theta = \pi/2$, it follows that the uniform pressure in the wake, P_w, is related to P_∞ by

$$P_\infty - P_w = \frac{5}{8}\rho U^2$$

Since the drag, D, is defined as the force acting on the surface in the direction of the uniform stream, it follows that

$$\begin{aligned} D &= \int_0^{\pi/2}(P_s - P_w)2\pi R^2\sin(\theta)\cos(\theta)d\theta \\ &= 2\pi R^2\int_0^{\pi/2}\left(\frac{5}{8}\rho U^2 + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)\right)\sin(\theta)\cos(\theta)d\theta \\ &= \frac{9}{16}\rho U^2\pi R^2 \end{aligned}$$

The drag coefficient, C_D, is defined as the drag divided by the frontal projected area of the body πR^2 and by $0.5\rho U_\infty^2$ and therefore becomes

$$C_D = \frac{9}{8}$$

Note that the actual drag coefficient of a sphere is substantially smaller than this because the pressure in the wake is normally much larger than given by the above conditions.