Solution to Problem 120P

We choose a coordinate system in which $y = 0$ is along the wall and $x = 0$ at the center of the vortices. To create the desired potential flow, the following velocity potential must be summed:

- Uniform stream, $\phi = Ux$
- 2-D counterclockwise potential vortex at $y = h$, $\phi = \frac{\Gamma}{2\pi}\theta_1$
- 2-D clockwise potential vortex at $y = -h$, $\phi = -\frac{\Gamma}{2\pi}\theta_2$

where

$$\theta_1 = \arctan\left(\frac{y-h}{x}\right) \quad \text{and} \quad \theta_2 = \arctan\left(\frac{y+h}{x}\right)$$

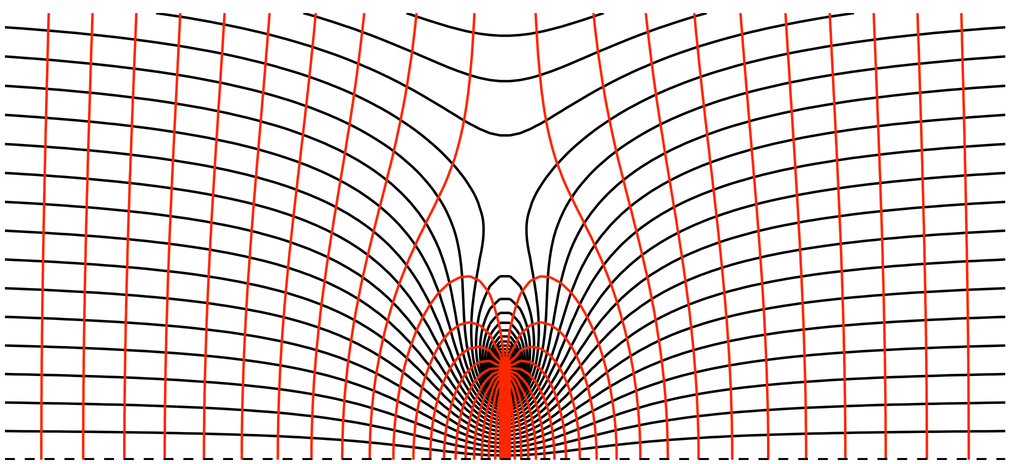
The combined velocity potential is

$$\phi = Ux + \frac{\Gamma}{2\pi}\arctan\left(\frac{y-h}{x}\right) - \frac{\Gamma}{2\pi}\arctan\left(\frac{y+h}{x}\right)$$

The corresponding streamfunction, ψ, is

$$\psi = Uy - \frac{\Gamma}{2\pi}\ln\left(\sqrt{(y-h)^2+x^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\sqrt{(y+h)^2+x^2}\right)$$

and therefore the streamlines (lines of constant ψ) are of the form



where the streamlines are shown in black and the flow proceeds from left to right (the equipotentials are shown in red).

The velocity component in the x-direction along the line $y = 0$ is then

$$u|_{y=0} = \left.\frac{\partial\phi}{\partial x}\right|_{y=0} = U + \frac{\Gamma h}{\pi(x^2+h^2)}$$

and, in accord with the zero normal velocity at the wall, the y-component of the velocity is zero. Note also that the x-component of the velocity far away from the vortex is equal to the free stream velocity ($u|_{x=\pm\infty,y=0} = U$). The pressure difference across the wall can be calculated using Bernoulli's equation,

$$P_\infty + \frac{1}{2}\rho U_\infty^2 = P_w + \frac{1}{2}\rho u|_{y=0}^2$$

where P_∞ is the pressure far upstream, far downstream and on the underside of the wall and P_w is the pressure on the upper side of the wall. It follows that

$$P_\infty - P_w = \frac{1}{2}\rho\left(u|_{y=0}^2 - U^2\right) = \frac{\rho h\Gamma}{\pi(x^2+h^2)}\left(U + \frac{h\Gamma}{2\pi(x^2+h^2)}\right)$$

The total force in the y-direction on the wall is given by the pressure difference integrated from $x = [-\infty, \infty]$

$$\begin{aligned}
F_y &= \int_{-\infty}^{\infty} (P_\infty - P_w)dx \\
&= \int_{-\infty}^{\infty} \frac{\rho h\Gamma}{\pi(x^2+h^2)}\left(U + \frac{h\Gamma}{2\pi(x^2+h^2)}\right) dx \\
&= \rho U\Gamma + \frac{1}{4}\frac{\rho\Gamma^2}{\pi h}
\end{aligned}$$

Thus, the lift due to the circulation ($\rho U\Gamma$) is modified by the term $0.25\rho\Gamma^2/\pi h$. Note also that the force in the x-direction is equal to zero:

$$F_x = \int_{-\infty}^{\infty} \delta P y dx = 0$$