Solution to Problem 116Ex

Given the following planar flow of an incompressible fluid:

$$\psi = Axyt$$

it follows that

$$u = \frac{\partial \psi}{\partial y} = Axt \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -Ayt$$

Then, since $\partial u/\partial y = 0$ and $\partial v/\partial x = 0$ and since

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

it follows that $\omega = 0$ and the flow is irrorational.

Since the flow is incompressible, inviscid, and irrotational, Bernoulli's equation applies. Bernoulli's equation for an *unsteady* flow is:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|u|^2 + gy = constant$$

To find ϕ for this equation, we integrate u and v:

$$\begin{aligned}
u = \frac{\partial \phi}{\partial x} &= Axt \\
\rightarrow \ \phi &= \frac{Ax^2t}{2} + c(y) \\
v = \frac{\partial \phi}{\partial y} &= c'(y) = -Ayt \\
\rightarrow \ c(y) &= -\frac{Ay^2t}{2} + c \\
\therefore \ \phi &= \frac{At}{2}(x^2 - y^2)
\end{aligned}$$

where c is an arbitrary constant. Therefore,

$$\frac{\partial \phi}{\partial t} = \frac{A}{2}(x^2 - y^2)$$

To find $\frac{1}{2}|u|^2$:

$$\begin{aligned}
\frac{1}{2}|u|^2 &= \frac{1}{2}(u^2 + v^2) \\
&= \frac{1}{2}(A^2x^2t^2 + A^2y^2t^2)
\end{aligned}$$

Substituting the above expressions into the Bernoulli equation:

$$\begin{aligned}
\frac{p}{\rho} &= constant - gy - \frac{\partial \phi}{\partial t} - \frac{1}{2}|u|^2 \\
&= constant - gy - \frac{A^2t^2}{2}(x^2 + y^2) - \frac{A}{2}(x^2 - y^2)
\end{aligned}$$

Setting $p = p_0$ at the origin determines that the $constant = p_0$ and

$$\frac{p}{\rho} = \frac{p_0}{\rho} - gy - \frac{A^2t^2}{2}(x^2 + y^2) - \frac{A}{2}(x^2 - y^2)$$