Solution to Problem 116B

Purely radial flow requires that $u_\theta = u_\phi = 0$ and that $\partial/\partial\theta = 0$ and $\partial/\partial\phi = 0$. Moreover the continuity equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

Since the flow is purely radial, this reduces to:

$$\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

Integrating with respect to r:

$$r^2\ u_r = f(t)$$

Since at $r = R(t)$, $u_r = dR/dt$ it follows that

$$f(t) = R^2\frac{dR}{dt} \quad \text{and} \quad u_r = \frac{R^2}{r^2}\frac{dR}{dt}$$

For purely radial flow, Euler's equations in the θ and ϕ directions are automatically satisfied. In the r direction, Euler's equation reduces to:

$$\rho\frac{Du_r}{Dt} = \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r}\right) = -\frac{\partial p}{\partial r}$$

Substituting the expression derived for u_r:

$$-\frac{\partial p}{\partial r} = \rho\left(\left[\frac{2R}{r^2}\left(\frac{dR}{dt}\right)^2 + \frac{R^2}{r^2}\frac{d^2R}{dt^2}\right] + \frac{R^2}{r^2}\frac{dR}{dt}\left[-2\frac{R^2}{r^3}\frac{dR}{dt}\right]\right)$$

Separating and integrating:

$$\int \partial p = \int -\rho\left(\frac{1}{r^2}\left[2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2}\right] - 2\frac{R^4}{r^5}\left(\frac{dR}{dt}\right)^2\right)\partial r$$

and therefore

$$p(r,t) = \rho\left(\frac{1}{r}\left[2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2}\right] - \frac{1}{2}\frac{R^4}{r^4}\left(\frac{dR}{dt}\right)^2\right) + c(t)$$

The unknown integration function $c(t)$ is evaluated knowing the pressure as $r \to \infty$, which is denoted by p_∞:

$$p(r \to \infty, t) = c(t) = p_\infty$$

and therefore

$$p(r,t) = \rho\left(\frac{1}{r}\left[2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2}\right] - \frac{1}{2}\frac{R^4}{r^4}\left(\frac{dR}{dt}\right)^2\right) + p_\infty$$

The pressure inside the bubble, p_B, equal to $p(r,t)$ evaluated at $r = R$, then completes the application of the boundary conditions:

$$p_B = p(R,t) = \rho\left[2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} - \frac{1}{2}\left(\frac{dR}{dt}\right)^2\right] + p_\infty$$

and therefore

$$p_B - p_\infty = \rho\left[\frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2}\right]$$

This is known as the Rayleigh equation for bubble growth and collapse; it connects the pressures at infinity and inside the bubble with the radius/time history, $R(t)$.