Solution to Problem 115D

You are given the following streamfunction for a planar incompressible flow:

$$\psi = Ur\left(1 - \frac{r_0^2}{r^2}\right)\sin\theta$$

where U and r_0 are constants and r, θ are polar coordinates. The velocities, given by the derivatives of the streamfunction are therefore

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}Ur\left(1 - \frac{r_0^2}{r^2}\right)\cos\theta = U\left(1 - \frac{r_0^2}{r^2}\right)\cos\theta$$

$$u_\theta = -\left[U\left(1 - \frac{r_0^2}{r^2}\right) + Ur\left(2\frac{r_0^2}{r^3}\right)\right]\sin\theta = -U\left(1 + \frac{r_0^2}{r^2}\right)\sin\theta$$

a) The velocities on a circle of radius r_0 are:

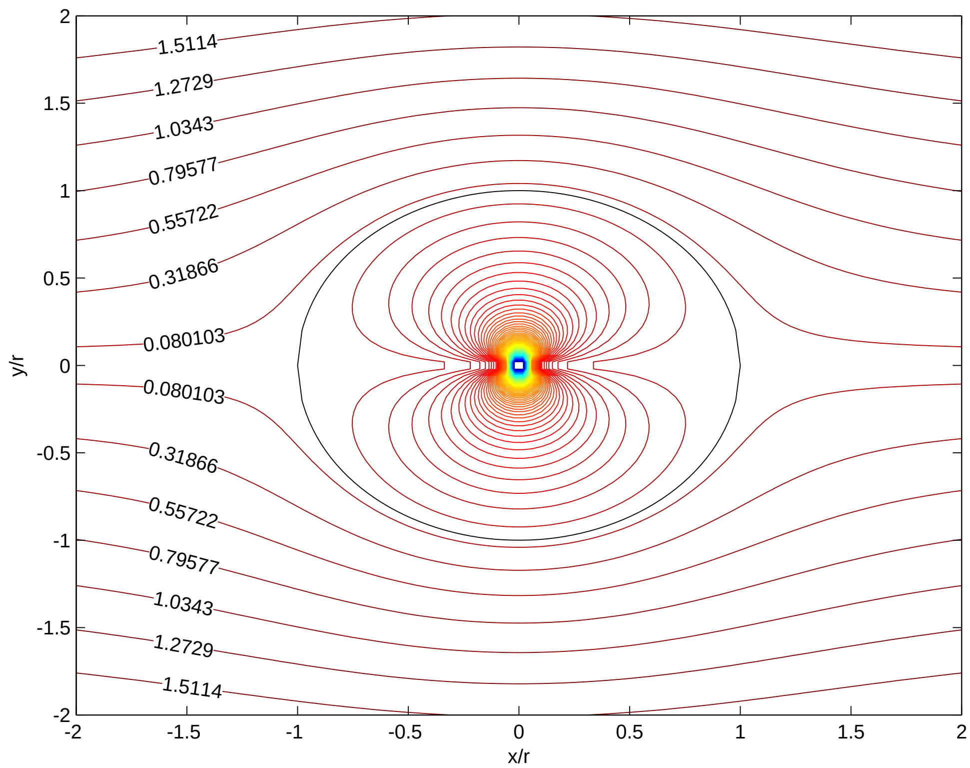
$$u_r|_{r=r_0} = 0$$

$$u_\theta|_{r=r_0} = -2U\sin\theta$$

and since there is no velocity normal to the circle $r = r_0$ this must be a streamline. From the expression for ψ it is the streamline with $\psi = 0$.

b) In addition from the expression for ψ we note that on the lines $\theta = 0$, $r > r_0$, and $\theta = \pi$, $r > r_0$ the streamfunction $\psi = 0$ and these lines are therefore part of the same streamline.

c) The sktech below shows the form of some of the other streamlines for $\psi > 0$.



d) For $r \gg r_0$ it follows that $u_r \to U\cos\theta$ and $u_\theta \to -U\sin\theta$ and consequently the magnitude of the flow velocity is $|\vec{u}| = \sqrt{u_r^2 + u_\theta^2} = U$ and the direction is in the $\theta = 0$ direction. Consequently the flow far away is a uniform stream of magnitude U in the $\theta = 0$ direction.

e) The streamfunction ψ represents the flow of a uniform stream of magnitude U around a stationary cylinder of radius r_0.