Solution to Problem 115C

Part (a)

The continuity equation for incompressible flow in vector form is

$$\nabla \cdot \vec{u} = 0$$

and for axisymmetric incompressible flow this becomes

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

Considering a stream function ψ defined by the relations

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r},$$

Substituting these definitions into the above continuity equation we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z} - \frac{1}{r}\frac{\partial^2 \psi}{\partial z \partial r} = 0$$

and therefore the above definition of ψ is correct for axisymmetric incompressible flow.

Part (b)

The continuity equation for compressible flow in vector form is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0$$

and for steady compressible planar flow this becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

Considering a stream function ψ defined by the relations

$$\rho u = \rho_o \frac{\partial \psi}{\partial y}, \qquad \rho v = -\rho_o \frac{\partial \psi}{\partial x}$$

where ρ_o is a constant.

Substituting these definitions into the above continuity equation we obtain

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = \rho_o \frac{\partial^2 \psi}{\partial x \partial y} - \rho_o \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

and therefore the above definition of ψ is correct for steady compressible planar flow.