Solution to Problem 108E

The shape of the pendant droplet will be governed by a balance of the pressure on either side of the surface with the curvature at that point:

$$p_I - p_O = \frac{S}{R}$$

where the pressure in the atmosphere is denoted by p_O (assumed uniform) and the pressure in the liquid is denoted by p_I which varies with the vertical elevation, y.

If we denote the half-breadth of the drop by $b(y)$ then the curvature at any height will be given by:

$$\frac{1}{R} = \frac{\frac{d^2b}{dy^2}}{\left[1 + \left(\frac{db}{dy}\right)^2\right]^{3/2}}$$

The last piece that we need is a description of how the pressure varies throughout the droplet. At the origin, the pressure will have some value, $\hat{p}$, different from the atmospheric pressure, p_A. As we move vertically through the droplet, the pressure will decrease hydrostatically giving:

$$p_I = \hat{p} - \rho g y$$

$$p_O = p_A$$

Combining all of this, the equation to determine the shape of the pendant droplet will be:

$$\frac{S\frac{d^2b}{dy^2}}{\left[1 + \left(\frac{db}{dy}\right)^2\right]^{3/2}} = (\hat{p} - p_A) - \rho g y$$

To nondimensionalize, we use:

$$b = b^\star \sqrt{\frac{S}{\rho g}} \qquad\qquad y = y^\star \sqrt{\frac{S}{\rho g}}$$

$$\frac{db}{dy} = \frac{db^\star}{dy^\star} \qquad\qquad \frac{d^2b}{dy^2} = \sqrt{\frac{\rho g}{S}}\frac{d^2b^\star}{dy^{\star 2}}$$

This leads to the following differential equation which needs to be solved to find $b(y)$: in which the single parameter is the first term on the right hand side:

$$\frac{\frac{d^2b^\star}{dy^{\star 2}}}{\left[1 + \left(\frac{db^\star}{dy^\star}\right)^2\right]^{3/2}} = \frac{(\hat{p} - p_A)}{\sqrt{\rho g S}} - y^\star$$

The solutions can be obtained numerically as a family of shapes with the single parameter $(\hat{p} - p_A)/\sqrt{\rho g S}$.