Problem 150L

In cylindrical coordinates, (r, θ, z), the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, μ, and density, ρ, are

$$\rho\left[\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right] = -\frac{\partial p}{\partial r} + f_r + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\rho\left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + f_z + \mu\,\nabla^2\,u_z$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinate directions, p is the pressure, f_r, f_θ, f_z are the body force components in the r, θ, z directions and the operators D/Dt and ∇^2 are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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Now consider the steady, planar, incompressible, viscous flow between two concentric cylinders. The inner cylinder has radius, a, and is rotating at an angular velocity, Ω (radians/second). The outer cylinder has radius, b, and is static. There is no flow in the direction parallel to the axis of the cylinders so only the velocity, u_θ, is non-zero. Body forces are to be neglected. The density of the fluid is denoted by ρ. Find:

- (a) The velocity distribution, $u_\theta(r)$, in the gap between the two cylinders.
- (b) The difference between the pressure on the outer surface of the inner cylinder and the pressure on the inner surface of the outer cylinder.

Note: The solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0 \qquad \text{is} \qquad y = A/x + Bx$$

where A and B are constants.