Problem 150J

In spherical coordinates, (r, θ, ϕ), the Navier-Stokes equations of motion for an incompressible fluid with uniform viscosity are:

$$\rho\left[\frac{Du_r}{Dt} - \frac{u_\theta^2 + u_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} + f_r + \mu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{2u_\theta \cot\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]$$

$$\rho\left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r} - \frac{u_\phi^2 \cot\theta}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin\theta \sin\theta} - \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]$$

$$\rho\left[\frac{Du_\phi}{Dt} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot\theta}{r}\right] = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + f_\phi + \mu\left[\nabla^2 u_\phi + \frac{2}{r^2 \sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2 \sin\theta \sin\theta} + \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]$$

where u_r, u_θ, u_ϕ are the velocities in the r, θ, ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

and the Laplacian operator is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta \sin\theta}\frac{\partial^2}{\partial^2 \phi}$$

Moreover, for an incompressible fluid the equation of continuity in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0$$

and for an incompressible, Newtonian fluid

$$\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} \quad ; \quad \sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

$$\sigma_{\phi\phi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}\right)$$

$$\sigma_{r\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \quad ; \quad \sigma_{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right)$$

$$\sigma_{\theta\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right)\right)$$

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An underwater explosion creates a purely radial flow ($u_\theta = u_\phi = 0$ and $\partial/\partial\theta = 0$ and $\partial/\partial\phi = 0$) in water surrounding a bubble whose radius, denoted by $R(t)$, is increasing with time. Assume that the water is incompressible. Derive the basic equation of bubble dynamics (the Rayleigh-Plesset equation) which is an ordinary differential equation connecting the bubble radius, $R(t)$, to the pressure in the bubble, p_b (assumed uniform), and the pressure in the liquid far from the bubble, p_∞. Neglect surface tension.