Problem 150I

In cylindrical coordinates, (r, θ, z), the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, μ, and density, ρ, are

$$\rho\left[\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right]=-\frac{\partial p}{\partial r}+f_r+\mu\left[\bigtriangledown^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\rho\left[\frac{Du_\theta}{Dt}+\frac{u_\theta u_r}{r}\right]=-\frac{1}{r}\frac{\partial p}{\partial \theta}+f_\theta+\mu\left[\bigtriangledown^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$\rho\frac{Du_z}{Dt}=-\frac{\partial p}{\partial z}+f_z+\mu\bigtriangledown^2 u_z$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinate directions, p is the pressure, f_r, f_θ, f_z are the body force components in the r, θ, z directions and the operators D/Dt and $\bigtriangledown^2$ are

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+u_z\frac{\partial}{\partial z}$$

$$\bigtriangledown^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

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Now consider the steady, planar, incompressible flow of a uniform stream of velocity, U, past a cylinder of radius, R, given by the following streamfunction, ψ:

$$\psi=Ur\left(1-\frac{R^2}{r^2}\right)\sin\theta$$

where r and θ are polar coordinates with origin in the center of the cylinder. Note that the velocities in the r and θ directions, denoted respectively by u_r and u_θ, are given by

$$u_r=\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad ; \quad u_\theta=-\frac{\partial \psi}{\partial r}$$

This is probably familiar to you as the solution to potential flow around a cylinder. In this problem show that this is also a solution to the Navier-Stokes equations provided that (1) you neglect the convective inertial terms and (2) ignore the fact that the solution does not satisfy the no-slip condition. Find the drag on the cylinder which this solution implies.

Find the drag on the cylinder which this solution implies. Note that in these polar coordinates in planar flow:

$$\sigma_{rr}=-p+2\mu\frac{\partial u_r}{\partial r} \quad ; \quad \sigma_{r\theta}=\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$$