Unsteady Stokes Flow

In order to elucidate some of the issues raised in the last section, it is instructive to examine solutions for the unsteady flow past a sphere in low Reynolds number Stokes flow. In the asymptotic case of zero Reynolds number, the solution of section (Nec) is unchanged by unsteadiness, and hence the solution at any instant in time is identical to the steady-flow solution for the same particle velocity. In other words, since the fluid has no inertia, it is always in static equilibrium. Thus the instantaneous force is identical to that for the steady flow with the same $V_i(t)$.

The next step is therefore to investigate the effects of small but nonzero inertial contributions. The Oseen solution provides some indication of the effect of the *convective* inertial terms, $u_j \partial u_i/\partial x_j$, in steady flow. Here we investigate the effects of the *unsteady* inertial term, $\partial u_i/\partial t$. Ideally it would be best to include *both* the $\partial u_i/\partial t$ term *and* the Oseen approximation to the convective term, $U\partial u_i/\partial x$. However, the resulting *unsteady* Oseen flow is sufficiently difficult that only small-time expansions for the impulsively started motions of droplets and bubbles exist in the literature (Pearcey and Hill 1956).

Consider, therefore the *unsteady* Stokes equations in the absence of the convective inertial terms:

$$\rho_C \frac{\partial u_i}{\partial t} = -\frac{\partial P}{\partial x_i} + \mu_C \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{Neh1}$$

Since both the equations and the boundary conditions used below are linear in u_i, we need only consider colinear particle and fluid velocities in one direction, say x_1. The solution to the general case of noncolinear particle and fluid velocities and accelerations may then be obtained by superposition. As in section (Neg) the colinear problem is solved by first transforming to an accelerating coordinate frame, x_i, fixed in the center of the particle so that $P = p + \rho_C x_1 dV/dt$. Elimination of P by taking the curl of equation (Neh1) leads to

$$(L - \frac{1}{\nu_C}\frac{\partial}{\partial t})L\psi = 0 \tag{Neh2}$$

where L is the same operator as defined in equation (Neg4). Guided by both the steady Stokes flow and the unsteady potential flow solution, one can anticipate a solution of the form

$$\psi = \sin^2\theta \ f(r,t) + \cos\theta \sin^2\theta \ g(r,t) + \cos\theta \ h(t) \tag{Neh3}$$

plus other spherical harmonic functions. The first term has the form of the steady Stokes flow solution; the last term would be required if the particle were a growing spherical bubble. After substituting equation (Neh3) into equation (Neh2), the equations for f, g, h are

$$(L_1 - \frac{1}{\nu_C}\frac{\partial}{\partial t})L_1 f = 0 \qquad \text{where} \qquad L_1 \equiv \frac{\partial^2}{\partial r^2} - \frac{2}{r^2} \tag{Neh4}$$

$$(L_2 - \frac{1}{\nu_C}\frac{\partial}{\partial t})L_2 g = 0 \qquad \text{where} \qquad L_2 \equiv \frac{\partial^2}{\partial r^2} - \frac{6}{r^2} \tag{Neh5}$$

$$(L_0 - \frac{1}{\nu_C}\frac{\partial}{\partial t})L_0 h = 0 \qquad \text{where} \qquad L_0 \equiv \frac{\partial^2}{\partial r^2} \tag{Neh6}$$

Moreover, the form of the expression for the force, F_1, on the spherical particle (or bubble) obtained by evaluating the stresses on the surface and integrating is

$$\frac{F_1}{\frac{4}{3}\rho_C \pi R^3} = \frac{dV}{dt} + \left\{ \frac{1}{r}\frac{\partial^2 f}{\partial r \partial t} + \frac{\nu_C}{r}\left(\frac{2}{r^2}\frac{\partial f}{\partial r} + \frac{2}{r}\frac{\partial^2 f}{\partial r^2} - \frac{\partial^3 f}{\partial r^3} \right) \right\}_{r=R} \tag{Neh7}$$

It transpires that this is *independent* of g or h. Hence only the solution to equation (Neh4) for $f(r,t)$ need be sought in order to find the force on a spherical particle, and the other spherical harmonics that might have been included in equation (Neh3) are now seen to be unnecessary.

Fourier or Laplace transform methods may be used to solve equation (Neh4) for $f(r,t)$, and we choose Laplace transforms. The Laplace transforms for the relative velocity $W(t)$, and the function $f(r,t)$ are denoted by $\hat{W}(s)$ and $\hat{f}(r,s)$:

$$\hat{W}(s) = \int_0^\infty e^{-st} W(t) dt \quad ; \quad \hat{f}(r,s) = \int_0^\infty e^{-st} f(r,t) dt \tag{Neh8}$$

Then equation (Neh4) becomes

$$(L_1 - \xi^2) L_1 \hat{f} = 0 \tag{Neh9}$$

where $\xi = (s/\nu_C)^{\frac{1}{2}}$, and the solution after application of the condition that $\hat{u}_1(s,t)$ far from the particle be equal to $\hat{W}(s)$ is

$$\hat{f} = -\frac{\hat{W} r^2}{2} + \frac{A(s)}{r} + B(s)(\frac{1}{r} + \xi) e^{-\xi r} \tag{Neh10}$$

where A and B are functions of s whose determination requires application of the boundary conditions on $r = R$. In terms of A and B the Laplace transform of the force $\hat{F}_1(s)$ is

$$\frac{\hat{F}_1}{\frac{4}{3}\rho_C \pi R^3} = \frac{d\hat{V}}{dt} + \left\{ \frac{s}{r}\frac{\partial \hat{f}}{\partial r} + \frac{\nu_C}{R}\left(-\frac{4\hat{W}}{r} + \frac{8A}{r^4} + CBe^{-\xi r} \right) \right\}_{r=R} \tag{Neh11}$$

where

$$C = \xi^4 + \frac{3\xi^3}{r} + \frac{3\xi^2}{r^2} + \frac{8\xi}{r^3} + \frac{8}{r^4} \tag{Neh12}$$

The classical solution (see Landau and Lifshitz 1959) is for a solid sphere (i.e., constant R) using the no-slip (Stokes) boundary condition for which

$$f(R,t) = \left. \frac{\partial f}{\partial r} \right|_{r=R} = 0 \tag{Neh13}$$

and hence

$$A = +\frac{\hat{W} R^3}{2} + \frac{3\hat{W} R \nu_C}{2s}\{1 + \xi R\} \quad ; \quad B = -\frac{3\hat{W} R \nu_C}{2s} e^{\xi R} \tag{Neh14}$$

so that

$$\frac{\hat{F}_1}{\frac{4}{3}\rho_C \pi R^3} = \frac{d\hat{V}}{dt} - \frac{3}{2}s\hat{W} - \frac{9\nu_C \hat{W}}{2R^2} - \frac{9\nu_C^{\frac{1}{2}}}{2R} s^{\frac{1}{2}} \hat{W} \tag{Neh15}$$

For a motion starting at rest at $t = 0$ the inverse Laplace transform of this yields

$$\frac{F_1}{\frac{4}{3}\rho_C \pi R^3} = \frac{dV}{dt} - \frac{3}{2}\frac{dW}{dt} - \frac{9\nu_C}{2R^2} W - \frac{9}{2R}(\frac{\nu_C}{\pi})^{\frac{1}{2}} \int_0^t \frac{dW(\tilde{t})}{d\tilde{t}} \frac{d\tilde{t}}{(t - \tilde{t})^{\frac{1}{2}}} \tag{Neh16}$$

where $\tilde{t}$ is a dummy time variable. This result must then be written in the original coordinate framework with $W = V - U$ and can be generalized to the noncolinear case by superposition so that

$$F_i = -\frac{1}{2} v \rho_C \frac{dV_i}{dt^*} + \frac{3}{2} v \rho_C \frac{dU_i}{dt^*} + \frac{9 v \mu_C}{2R^2}(U_i - V_i) + \frac{9 v \rho_C}{2R}(\frac{\nu_C}{\pi})^{\frac{1}{2}} \int_0^{t^*} \frac{d(U_i - V_i)}{d\tilde{t}} \frac{d\tilde{t}}{(t^* - \tilde{t})^{\frac{1}{2}}} \tag{Neh17}$$

where d/dt^* is the Lagrangian time derivative following the particle. This is then the general force on the particle or bubble in unsteady Stokes flow when the Stokes boundary conditions are applied.

Compare this result with that obtained from the potential flow analysis, equation (Neg18) with v taken as constant. It is striking to observe that the coefficients of the added mass terms involving dV_i/dt^* and dU_i/dt^* are identical to those of the potential flow solution. On superficial examination it might be noted that dU_i/dt^* appears in equation (Neh17) whereas DU_i/Dt^* appears in equation (Neg18); the difference is, however, of order $W_j \partial U_i/dx_j$ and terms of this order have already been dropped from the equation of motion on the basis that they were negligible compared with the temporal derivatives like $\partial W_i/\partial t$. Hence it is inconsistent with the initial assumption to distinguish between d/dt^* and D/Dt^* in the present unsteady Stokes flow solution.

The term $9\nu_C W/2R^2$ in equation (Neh17) is, of course, the steady Stokes drag. The new phenomenon introduced by this analysis is contained in the last term of equation (Neh17). This is a fading memory term that is often named the Basset term after one of its identifiers (Basset 1888). It results from the fact that additional vorticity created at the solid particle surface due to relative acceleration diffuses into the flow and creates a temporary perturbation in the flow field. Like all diffusive effects it produces an $\omega^{\frac{1}{2}}$ term in the equation for oscillatory motion.

Before we conclude this section, comment should be included on three other analytical results. Morrison and Stewart (1976) considered the case of a spherical bubble for which the Hadamard-Rybczynski boundary conditions rather than the Stokes conditions are applied. Then, instead of the conditions of equation (Neh13), the conditions for zero normal velocity and zero shear stress on the surface require that

$$f(R,t) = \left\{ \frac{\partial^2 f}{\partial r^2} - \frac{2}{r}\frac{\partial f}{\partial r} \right\}_{r=R} = 0 \qquad \text{(Neh18)}$$

and hence in this case (see Morrison and Stewart 1976)

$$A(s) = +\frac{\hat{W}R^3}{2} + \frac{3\hat{W}R(1+\xi R)}{\xi^2(3+\xi R)} \;;\; B(s) = -\frac{3\hat{W}Re^{+\xi R}}{\xi^2(3+\xi R)} \qquad \text{(Neh19)}$$

so that

$$\frac{\hat{F}_1}{\frac{4}{3}\pi\rho_C R^3} = \frac{d\hat{V}}{dt} - \frac{9\hat{W}\nu_C}{R^2} - \frac{3}{2}\hat{W}s + \frac{6\nu_C\hat{W}}{R^2\left\{1 + s^{\frac{1}{2}}R/3\nu_C^{\frac{1}{2}}\right\}} \qquad \text{(Neh20)}$$

The inverse Laplace transform of this for motion starting at rest at $t = 0$ is

$$\frac{F_1}{\frac{4}{3}\rho_C\pi R^3} = \frac{dV}{dt} - \frac{3}{2}\frac{dW}{dt} - \frac{3\nu_C W}{R^2} \qquad \text{(Neh21)}$$

$$-\frac{6\nu_C}{R^2}\int_0^t \frac{dW(\tilde{t})}{d\tilde{t}} \exp\left\{\frac{9\nu_C(t-\tilde{t})}{R^2}\right\} \operatorname{erfc}\left\{\left(\frac{9\nu_C(t-\tilde{t})}{R^2}\right)^{\frac{1}{2}}\right\} d\tilde{t}$$

Comparing this with the solution for the Stokes conditions, we note that the first two terms are unchanged and the third term is the expected Hadamard-Rybczynski steady drag term (see equation (Nec6)). The last term is significantly different from the Basset term in equation (Neh17) but still represents a fading memory.

More recently, Magnaudet and Legendre (1998) have extended these results further by obtaining an expression for the force on a particle (bubble) whose radius is changing with time.

Another interesting case is that for unsteady Oseen flow, which essentially consists of attempting to solve the Navier-Stokes equations with the convective inertial terms approximated by $U_j \partial u_i / \partial x_j$. Pearcey and Hill (1956) have examined the small-time behavior of droplets and bubbles started from rest when this term is included in the equations.