Flow around a Sphere at Low Reynolds Number

At the other end of the Reynolds number spectrum is the classic Stokes solution for flow around a sphere. In this limit the terms on the left-hand side of equation (Nea2) are neglected and the viscous term retained. This solution has the form

$$\psi = \sin^2\theta \left\{ -\frac{Wr^2}{2} + \frac{A}{r} + Br \right\} \tag{Nec1}$$

$$u_r = \cos\theta \left\{ -W + \frac{2A}{r^3} + \frac{2B}{r} \right\} \tag{Nec2}$$

$$u_\theta = -\sin\theta \left\{ -W - \frac{A}{r^3} + \frac{B}{r} \right\} \tag{Nec3}$$

where A and B are constants to be determined from the boundary conditions on the surface of the sphere. The force, F, on the *particle* in the x_1 direction is

$$F_1 = \frac{4}{3}\pi R^2 \rho_C \nu_C \left\{ -\frac{4W}{R} + \frac{8A}{R^4} + \frac{2B}{R^2} \right\} \tag{Nec4}$$

Several subcases of this solution are of interest in the present context. The first is the classic Stokes (1851) solution for a solid sphere in which the no-slip boundary condition, $(u_\theta)_{r=R} = 0$, is applied (in addition to the kinematic condition $(u_r)_{r=R} = 0$). This set of boundary conditions, referred to as the Stokes boundary conditions, leads to

$$A = -\frac{WR^3}{4} \quad , \quad B = +\frac{3WR}{4} \qquad \text{and} \qquad F_1 = -6\pi\rho_C\nu_C WR \tag{Nec5}$$

The second case originates with Hadamard (1911) and Rybczynski (1911) who suggested that, in the case of a bubble, a condition of zero shear stress on the sphere surface would be more appropriate than a condition of zero tangential velocity, u_θ. Then it transpires that

$$A = 0 \quad , \quad B = +\frac{WR}{2} \qquad \text{and} \qquad F_1 = -4\pi\rho_C\nu_C WR \tag{Nec6}$$

Real bubbles may conform to either the Stokes or Hadamard-Rybczynski solutions depending on the degree of contamination of the bubble surface, as we shall discuss in more detail in section (Nfd). Finally, it is of interest to observe that the potential flow solution given in equations (Neb5) to (Neb8) is also a subcase with

$$A = +\frac{WR^3}{2} \quad , \quad B = 0 \qquad \text{and} \qquad F_1 = 0 \tag{Nec7}$$

However, another paradox, known as the Whitehead paradox, arises when the validity of these Stokes flow solutions at small (rather than zero) Reynolds numbers is considered. The nature of this paradox can be demonstrated by examining the magnitude of the neglected term, $u_j\partial u_i/\partial x_j$, in the Navier-Stokes equations relative to the magnitude of the retained term $\nu_C\partial^2 u_i/\partial x_j\partial x_j$. As is evident from equation (Nec1), far from the sphere the former is proportional to W^2R/r^2 whereas the latter behaves like $\nu_C WR/r^3$. It follows that although the retained term will dominate close to the body (provided the Reynolds number $Re = 2WR/\nu_C \ll 1$), there will always be a radial position, r_c, given by $R/r_c = Re$ beyond which the neglected term will exceed the retained viscous term. Hence, even if $Re \ll 1$, the Stokes solution is not uniformly valid. Recognizing this limitation, Oseen (1910) attempted to correct the Stokes solution by retaining in the basic equation an approximation to $u_j \partial u_i / \partial x_j$ that would be valid in the far field, $-W \partial u_i / \partial x_1$. Thus the Navier-Stokes equations are approximated by

$$-W \frac{\partial u_i}{\partial x_1} = -\frac{1}{\rho_C} \frac{\partial p}{\partial x_i} + \nu_C \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{Nec8}$$

Oseen was able to find a closed form solution to this equation that satisfies the Stokes boundary conditions approximately:

$$\psi = -WR^2 \left\{ \frac{r^2 \sin^2 \theta}{2R^2} + \frac{R \sin^2 \theta}{4r} + \frac{3\nu_C (1 + \cos\theta)}{2WR} \left(1 - e^{\frac{Wr}{2\nu_C (1-\cos\theta)}} \right) \right\} \tag{Nec9}$$

which yields a drag force

$$F_1 = -6\pi \rho_C \nu_C W R \left\{ 1 + \frac{3}{16}\, Re \right\} \tag{Nec10}$$

It is readily shown that equation (Nec9) reduces to equation (Nec1) as $Re \to 0$. The corresponding solution for the Hadamard-Rybczynski boundary conditions is not known to the author; its validity would be more questionable since, unlike the case of Stokes boundary conditions, the inertial terms $u_j \partial u_i / \partial x_j$ are not identically zero on the surface of the bubble.

Proudman and Pearson (1957) and Kaplun and Lagerstrom (1957) showed that Oseen's solution is, in fact, the first term obtained when the method of matched asymptotic expansions is used in an attempt to patch together consistent asymptotic solutions of the full Navier-Stokes equations for both the near field close to the sphere and the far field. They also obtained the next term in the expression for the drag force.

$$F_1 = -6\pi \rho_C \nu_C W R \quad \left\{ 1 + \frac{3}{16} Re + \frac{9}{160} Re^2 ln \left(\frac{Re}{2} \right) + 0(Re^2) \right\} \tag{Nec11}$$

The additional term leads to an error of 1% at $Re = 0.3$ and does not, therefore, have much practical consequence.

The most notable feature of the Oseen solution is that the geometry of the streamlines depends on the Reynolds number. The downstream flow is *not* a mirror image of the upstream flow as in the Stokes or potential flow solutions. Indeed, closer examination of the Oseen solution reveals that, downstream of the sphere, the streamlines are further apart and the flow is slower than in the equivalent upstream location. Furthermore, this effect increases with Reynolds number. These features of the Oseen solution are entirely consistent with experimental observations and represent the initial development of a wake behind the body.

The flow past a sphere at Reynolds numbers between about 0.5 and several thousand has proven intractable to analytical methods though numerical solutions are numerous. Experimentally, it is found that a recirculating zone (or vortex ring) develops close to the rear stagnation point at about $Re = 30$ (see Taneda 1956 and figure 1). With further increase in the Reynolds number this recirculating zone or wake expands. Defining locations on the surface by the angle from the front stagnation point, the separation point moves forward from about 130° at $Re = 100$ to about 115° at $Re = 300$. In the process the wake reaches a diameter comparable to that of the sphere when $Re \approx 130$. At this point the flow becomes unstable and the ring vortex that makes up the wake begins to oscillate (Taneda 1956). However, it continues to be attached to the sphere until about $Re = 500$ (Torobin and Gauvin 1959).

At Reynolds numbers above about 500, vortices begin to be shed and then convected downstream. The frequency of vortex shedding has not been studied as extensively as in the case of a circular cylinder and seems to vary more with Reynolds number. In terms of the conventional Strouhal number, Str, defined as

$$Str = 2fR/W \tag{Nec12}$$

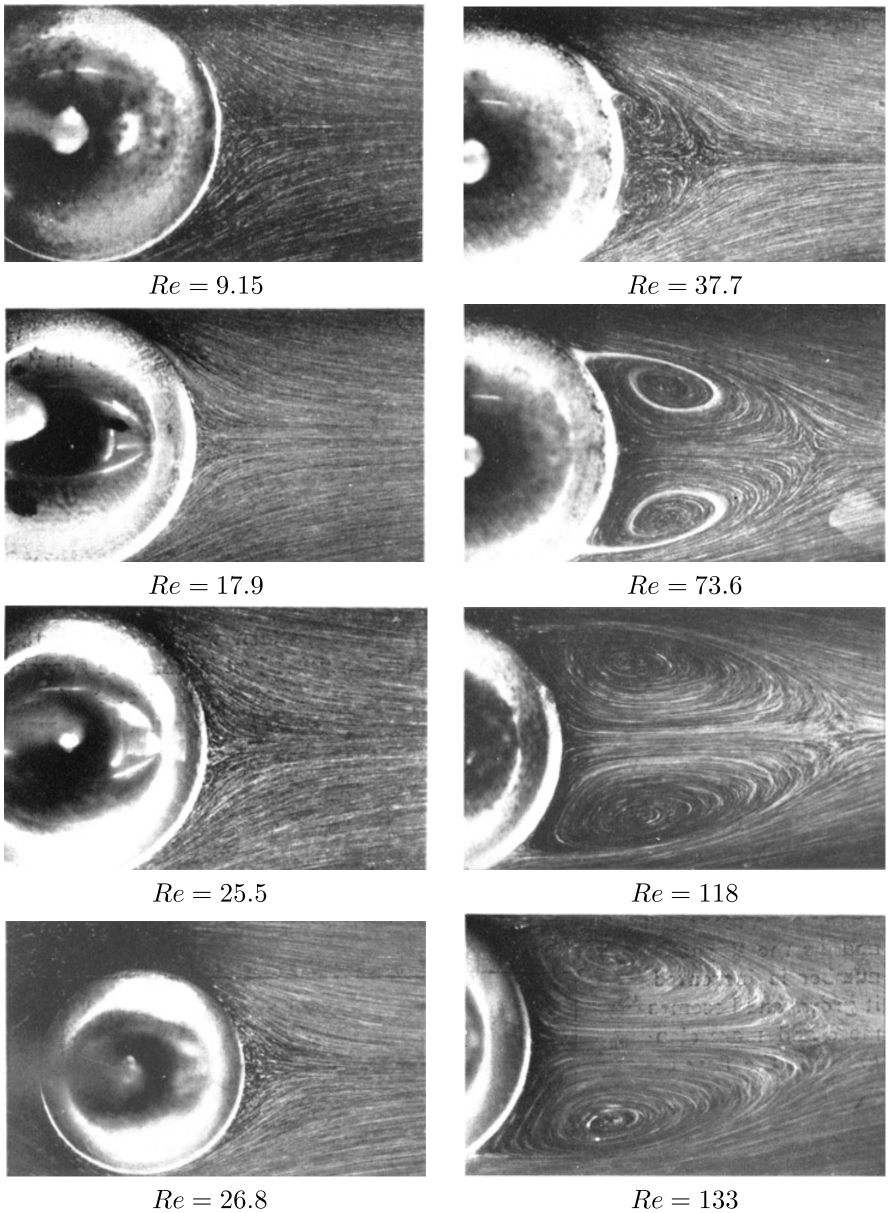

Figure 1: Streamlines of steady flow (from left to right) past a sphere at various Reynolds numbers (from Taneda 1956, reproduced by permission of the author).

the vortex shedding frequencies, f, that Moller (1938) observed correspond to a range of Str varying from 0.3 at $Re = 1000$ to about 1.8 at $Re = 5000$. Furthermore, as Re increases above 500 the flow develops a fairly steady *near-wake* behind which vortex shedding forms an unsteady and increasingly turbulent *far-wake.* This process continues until, at a value of Re of the order of 1000, the flow around the sphere and in the near-wake again becomes quite steady. A recognizable boundary layer has developed on the front of the sphere and separation settles down to a position about 84° from the front stagnation point. Transition to turbulence occurs on the free shear layer (which defines the boundary of the near-wake) and moves progressively forward as the Reynolds number increases. The flow is similar to that of the top photograph in section (Neb). Then the events described in the previous section occur with further increase in the Reynolds number.

Since the Reynolds number range between 0.5 and several hundred can often pertain in multiphase flows, one must resort to an empirical formula for the drag force in this regime. A number of empirical results are available; for example, Klyachko (1934) recommends

$$F_1 = -6\pi\rho_C\nu_C W R \left\{ 1 + \frac{Re^{\frac{2}{3}}}{6} \right\} \tag{Nec13}$$

which fits the data fairly well up to $Re \approx 1000$. At $Re = 1$ the factor in the square brackets is 1.167, whereas the same factor in equation (Nec10) is 1.187. On the other hand, at $Re = 1000$, the two factors are respectively 17.7 and 188.5.